

FROM FEYNMAN-KAC FORMULAE TO NUMERICAL STOCHASTIC HOMOGENIZATION IN ELECTRICAL IMPEDANCE TOMOGRAPHY

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ABSTRACT. In this paper, we use the theory of symmetric Dirichlet forms to derive Feynman-Kac formulae for the forward problem of electrical impedance tomography with possibly anisotropic, merely measurable conductivities corresponding to different electrode models on bounded Lipschitz domains. Subsequently, we employ these Feynman-Kac formulae to rigorously justify stochastic homogenization in the case of a stochastic boundary value problem arising from an inverse anomaly detection problem. Motivated by this theoretical result, we prove an estimate for the speed of convergence of the projected mean-square displacement of the underlying process which may serve as the theoretical foundation for the development of new scalable stochastic numerical homogenization schemes.

1. INTRODUCTION

Electrical impedance tomography (EIT) aims to reconstruct the unknown conductivity κ in the conductivity equation

$$(1) \quad \nabla \cdot (\kappa \nabla u) = 0 \quad \text{in } D$$

from current and voltage measurements on the boundary of the domain D . This inverse conductivity problem is known to be severely ill-posed, that is, its solution is extremely sensitive with respect to measurement and modeling errors. As a result, EIT suffers from inherent low resolution and due to this limitation, many practical applications focus on the detection of conductivity anomalies in a known background conductivity rather than conductivity imaging. In the mathematical modeling of such inverse anomaly detection problems, randomness typically reflects a lack of precise information about the meso- and microstructure of the heterogeneous background conductivity, which may fluctuate on many scales. Recently, the second author has proposed a novel method for the detection of conductivity anomalies in a random background conductivity which is based on homogenization of the underlying stochastic boundary value problem, cf. [55].

Although the homogenization theory for elliptic divergence form operators is well-developed, cf., e.g., [6, 47, 48, 64], the numerical approximation of the *effective conductivity* in the random setting still poses major challenges. The commonly used deterministic methods based on a discretization of the so-called *auxiliary problem* have two main drawbacks. First, the auxiliary problem is formulated on the whole space \mathbb{R}^d and second, it has to be solved for almost every realization of the random medium. That is, truncations of the auxiliary problem have to be considered and choosing an

appropriate spatial truncation with appropriate boundary conditions is a delicate issue, cf. [11]. Moreover, in practically relevant cases, such as high contrast digitized random media, it is extremely difficult to solve the corresponding variational problems by usual deterministic methods, such as the finite element or the finite difference method, due to the behavior of the solutions near the corner points. Therefore, numerical approximation of the effective conductivity can be prohibitively expensive in terms of computation time. As a matter of fact, practitioners often choose to avoid these computations at all and rather content themselves with theoretical bounds, cf., e.g., [60]. However, it has been reported in the physical literature that the shortcomings of the standard deterministic methods can be circumvented by using continuum microscale Monte Carlo simulation of certain diffusion processes evolving in random media instead, cf., e.g., [36, 56, 61, 38, 39]. In this work we give a rigorous mathematical justification for homogenizing the EIT forward problem using such methods by studying the interconnection between reflecting diffusion processes and certain boundary value problems for the conductivity equation. More precisely, we derive *Feynman-Kac formulae* for solutions of the deterministic conductivity equation (1) posed on a bounded domain $D \subset \mathbb{R}^d$, $d \geq 2$, with Lipschitz boundary ∂D and possibly anisotropic uniformly elliptic and uniformly bounded conductivity subject to different boundary conditions modeling electrode measurements. Subsequently, we employ the Feynman-Kac formulae to prove a homogenization result for the corresponding stochastic boundary value problem which justifies the use of stochastic numerical homogenization schemes based on simulation of the underlying diffusion processes in order to approximate the effective conductivity. Finally, we prove an estimate for the speed of convergence of the projected mean-square displacement of the underlying diffusion processes. The main advantage of the presented approach to numerical homogenization, beside its inherent parallelism, is that its convergence rate is dimension-independent and its computational cost grows only linearly with the dimension.

It is well known, that reflecting diffusion processes generated by non-divergence form operators with smooth coefficients on bounded, smooth domains, are *Feller processes* satisfying Skorohod type stochastic differential equations. The construction in the case of divergence form operators with merely measurable coefficients requires the theory of *symmetric Dirichlet forms* which has its origin in the *energy method* used by Dirichlet to address the boundary value problem in classical electrostatics that was subsequently named after him. When D is a bounded Lipschitz domain, Bass and Hsu [5] constructed the reflecting Brownian motion living on \overline{D} by showing that the so-called *Martin-Kuramochi boundary* coincides with the Euclidean boundary in this case. A general diffusion process on a bounded Lipschitz domain, even allowing locally a finite number of Hölder cusps, was first constructed by Fukushima and Tomisaki [26]. In this work, we use such a Dirichlet form construction in order to derive Feynman-Kac representation formulae for the solutions of Neumann, respectively Robin, boundary value problems modeling EIT measurements. Probabilistic approaches to both, parabolic and elliptic boundary value problems for second order differential operators have been studied by many authors, starting with Feynman's Princeton thesis [20] and the article [34] by Kac. The probabilistic approach to the Dirichlet problem for a general class of second-order elliptic operators with merely measurable coefficients, even allowing singularities of a certain type, was elaborated by Chen and Zhang [14]; see also Zhang's paper [63]. However, there are only few works that treat Feynman-Kac representation formulae for Neumann or Robin type boundary conditions. Moreover, the

approaches existing in the literature consider either the Laplacian, see, e.g., [5, 9, 31], or non-divergence form operators with smooth coefficients, see, e.g., [22, 49, 10]. For the particular case of the conductivity equation on a bounded Lipschitz domain, we generalize both, the Feynman-Kac formula for the Robin problem on domains with boundary of class C^3 for an isotropic $C^{2,\gamma}$ -smooth conductivity, $\gamma > 0$, obtained by Papanicolaou [49] as well as the representation obtained by Benchérif-Madani and Pardoux [10] for the Neumann problem under similar regularity assumptions. While both of the aforementioned approaches use stochastic differential equations and Itô calculus, our approach is based on the theory of symmetric Dirichlet forms, following the pioneering work [5] for the reflecting Brownian motion by Bass and Hsu. We derive in this work Feynman-Kac formulae for both, the Robin boundary value problem, corresponding to the so-called *complete electrode model*, as well as the Neumann boundary value problem, corresponding to the so-called *continuum model*. Both formulae are valid for possibly anisotropic, uniformly elliptic and uniformly bounded conductivities with merely measurable coefficients on bounded Lipschitz domains. During the preparation of this work we became aware of the paper [15] by Chen and Zhang, where a probabilistic approach to some mixed boundary value problems with singular coefficients is derived. In contrast to our setting, however, the mixed boundary condition studied there results from a singular lower-order term of the differential operator.

Homogenization of reflected stochastic differential equations and partial differential equations with Neumann boundary conditions, respectively, in half-space type domains have been studied for periodic coefficients in [4, 6, 59] and for random divergence form operators with smooth coefficients in [52]. In contrast to boundary value problems with homogeneous Dirichlet boundary conditions, these problems are non-translation invariant, which excludes the standard stochastic homogenization approach via the so-called *environment as viewed from the particle*. Employing the Feynman-Kac formula in conjunction with a recently obtained invariance principle for reflecting diffusion processes associated with random divergence form operators with merely measurable coefficients due to Chen, Croydon and Kumagai [13] we provide a homogenization result for a stochastic forward problem built on the complete electrode model. Clearly, such a result motivates the derivation of stochastic numerical homogenization schemes for the approximation of the effective conductivity which are based on simulation of the underlying diffusion processes. However, the convergence analysis of such a method requires a quantitative convergence result that is stronger than the usual qualitative results obtained from the central limit theorem for martingales. As in the case of a discrete random walk in random environment, cf. Gloria and Mourrat [27], it turns out that the behavior at the bottom of the spectrum of the infinitesimal generator of the environment as viewed from the particle process, projected on a suitably chosen function, yields bounds on the approximation error. This spectral behavior has been the subject of recent interest. Most notably, Gloria, Neukamm and Otto [28] have obtained optimal estimates in the discrete case which have been carried over to the continuum case by Gloria and Otto [29]. The main difficulty in obtaining such estimates for diffusion processes evolving in random media arises from the lack of a Poincaré inequality for the horizontal derivative in the space of square integrable functions on the probability space which corresponds to the random medium. Therefore, in contrast to the periodic case, where the Poincaré inequality on the torus is available, one can not expect a spectral gap in the random case. Still, it has been shown that the bottom of the spectrum is sufficiently “thin”. Using these estimates together with a classical

argument due to Kipnis and Varadhan [40], we obtain an estimate for the speed of convergence of the projected mean-square displacement of the underlying diffusion process in a random medium to its limit. Qualitative results of this kind have been obtained by Kipnis and Varadhan in the case of discrete random walks in random environments and by De Masi, Ferrari, Goldstein and Wick [18] in the continuum case, whereas qualitative results in the case of discrete random walks have been proved more recently by Gloria and Mourrat [27] and Egloff, Gloria, Mourrat and Nguyen [19]. Finally, we refer to the paper [44] by Mourrat which initiated the idea of using the Kipnis and Varadhan argument in order to obtain quantitative results.

The rest of the paper is structured as follows: We start in Section 2 by briefly introducing our notation. In Section 3, we recall the modeling of electrode measurements in EIT as well as the modeling of random heterogeneous media. Moreover, we introduce the stochastic forward problem we are interested in. In Section 4, we describe the construction of reflecting diffusion processes via Dirichlet form theory and in Section 5 we derive Skorohod decompositions for two practically relevant classes of conductivities. Subsequently, in Section 6, the Feynman-Kac formulae for the deterministic boundary value problems will be derived. Then in Section 7 we study the interconnection between Feynman-Kac formulae, stochastic homogenization and stochastic numerics. Finally, we conclude with a brief summary of our results.

2. NOTATION

Let D denote a bounded Lipschitz domain in \mathbb{R}^d , $d \geq 2$, with connected complement and Lipschitz parameters (r_D, c_D) , i.e., for every $x \in \partial D$ we have after rotation and translation that $\partial D \cap B(x, r_D)$ is the graph of a Lipschitz function in the first $d-1$ coordinates with Lipschitz constant no larger than c_D and $D \cap B(x, r_D)$ lies above the graph of this function. Moreover, we set $\mathbb{R}_-^d := \{x \in \mathbb{R}^d : x \cdot \nu < 0\}$, with $\nu = e_d$ the outward unit normal on \mathbb{R}^{d-1} , where we identify the boundary of \mathbb{R}_-^d with \mathbb{R}^{d-1} , with straightforward abuse of notation.

For Lipschitz domains, there exists a unique outward unit normal vector ν a.e. on ∂D so that the real Lebesgue spaces $L^p(D)$ and $L^p(\partial D)$ can be defined in the standard manner with the usual L^p norms $\|\cdot\|_p$, $p = 1, 2, \infty$. The standard L^2 inner-products are denoted by $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\partial D}$, respectively. The d -dimensional Lebesgue measure is denoted by m , the $(d-1)$ -dimensional Lebesgue surface measure is denoted by σ and $|\cdot|$ denotes the Euclidean norm on \mathbb{R}^d .

By $(\Gamma, \mathcal{G}, \mathcal{P})$ we always mean a complete probability space corresponding to a random medium. We use the notation ω for an arbitrary element of Γ and \mathbb{M} for the expectation with respect to the probability measure \mathcal{P} . We use bold letters to denote functions on $(\Gamma, \mathcal{G}, \mathcal{P})$, while we use italic letters for the corresponding realizations on $\mathbb{R}^d \times \Gamma$. The canonical probability space corresponding to diffusion processes evolving in a deterministic medium starting in x is denoted $(\Omega, \mathcal{F}, \mathbb{P}_x)$ and the expectation with respect to \mathbb{P}_x is denoted \mathbb{E}_x . If the process is evolving in a random medium, we indicate this with a superscript ω for the probability measure, i.e., the measure \mathbb{P}_x^ω corresponds to the particular realization ω of the medium. Finally, the product probability space corresponding to the *annealed measure* $\overline{\mathbb{P}} := \mathcal{P} \mathbb{P}_0^\omega$ on $\overline{\Omega} := \Gamma \times \Omega$, which is obtained by integrating with respect to the measure \mathbb{P}_0^ω and subsequent averaging over the realizations of the random medium, is denoted $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. The expectation with respect to $\overline{\mathbb{P}}$ is denoted $\overline{\mathbb{E}}$.

All functions in this work will be real-valued and derivatives are understood in distributional sense. We use a diamond subscript to denote subspaces of the standard Sobolev spaces containing functions with vanishing mean and interpret integrals over ∂D as dual evaluations with a constant function, if necessary. For example, we will frequently use the spaces

$$H_{\diamond}^{\pm 1/2}(\partial D) := \left\{ \phi \in H^{\pm 1/2}(\partial D) : \langle \phi, 1 \rangle_{\partial D} = 0 \right\}$$

and

$$H_{\diamond}^1(D) := \left\{ \phi \in H^1(D) : \langle \phi, 1 \rangle = 0 \right\}.$$

Moreover, we will frequently assume that ∂D is partitioned into two disjoint parts, $\partial_1 D$ and $\partial_2 D$. We denote by $H_0^1(D \cup \partial_1 D)$ the closure of $C_c^\infty(D \cup \partial_1 D)$, the linear subspace of $C^\infty(\overline{D})$ consisting of functions ϕ such that $\text{supp}(\phi)$ is a compact subset of $D \cup \partial_1 D$, in $H^1(D)$. Moreover, we define the Bochner space $L^2(\Gamma; H_0^1(D \cup \partial_1 D))$

$$= \left\{ \phi : \Gamma \rightarrow H_0^1(D \cup \partial_1 D) : \int_{\Gamma} \|\phi(\cdot, \omega)\|_{H_0^1(D \cup \partial_1 D)}^2 d\mathcal{P}(\omega) < \infty \right\},$$

see, e.g., [3] for properties of this space.

For the reason of notational compactness, we use the Iverson brackets: Let S be a mathematical statement, then

$$[S] = \begin{cases} 1, & \text{if } S \text{ is true} \\ 0, & \text{otherwise.} \end{cases}$$

We also use the Iverson brackets $[x \in B]$ to denote the indicator function of a set B , which we abbreviate by $[B]$ if there is no danger of confusion.

In what follows, all unimportant constants are denoted c , sometimes with additional subscripts, and they may vary from line to line.

3. ELECTRICAL IMPEDANCE TOMOGRAPHY FORWARD PROBLEMS

3.1. Modeling of electrode measurements. We assume that the, possibly anisotropic, conductivity is defined by a symmetric, matrix-valued function $\kappa : D \rightarrow \mathbb{R}^{d \times d}$ with components in $L^\infty(D)$ such that κ is uniformly bounded and uniformly elliptic, i.e., there exists some constant $c > 0$ such that

$$(2) \quad c^{-1}|\xi|^2 \leq \xi \cdot \kappa(x)\xi \leq c|\xi|^2, \quad \text{for every } \xi \in \mathbb{R}^d \text{ and a.e. } x \in D.$$

The forward problem of electrical impedance tomography can be described by different measurement models. In the so-called *continuum model*, the conductivity equation (1) is equipped with a co-normal boundary condition

$$(3) \quad \partial_{\kappa\nu} u := \kappa\nu \cdot \nabla u|_{\partial D} = f \quad \text{on } \partial D,$$

where f is a measurable function modeling the signed density of the outgoing current. The boundary value problem (1), (3) has a solution if and only if

$$(4) \quad \langle f, 1 \rangle_{\partial D} = 0.$$

Physically speaking, this means that the current must be conserved. Given an appropriate function f , the solution to (1), (3) is unique up to an additive constant, which physically corresponds to the choice of the ground level of the potential. If $f \in H_{\diamond}^{-1/2}$,

then there exists a unique equivalence class of functions $u \in H^1(D)/\mathbb{R}$ that satisfies the weak formulation of the boundary value problem

$$\int_D \kappa \nabla u \cdot \nabla v \, dx = \langle f, v|_{\partial D} \rangle_{\partial D} \quad \text{for all } v \in H^1(D)/\mathbb{R},$$

where $v|_{\partial D} := \gamma v$ and $\gamma : H^1(D)/\mathbb{R} \rightarrow H^{1/2}(\partial D)/\mathbb{R} = (H_\diamond^{-1/2}(\partial D))'$ is the standard trace operator. Note that we occasionally write v instead of $v|_{\partial D}$ for the sake of readability.

In practical EIT measurement, a number of electrodes, denoted $E_1, \dots, E_N \subset \partial D$, are attached on the boundary of the object D . These electrodes are modeled by disjoint surface patches given by simply connected subsets of ∂D , each having a Lipschitz boundary curve. The most accurate forward model for real-life EIT is the so-called *complete electrode model* which takes into account the fact that during electrode measurements there is a *contact impedance* caused by a thin, highly resistive layer at the electrode object interface. It was demonstrated experimentally that the complete electrode model can correctly predict measurements up to instrument precision, cf. [57]. For a given voltage pattern $U \in \mathbb{R}^N$ the boundary conditions for the complete electrode model are given by

$$(5) \quad \kappa \nu \cdot \nabla u|_{\partial D} + g u|_{\partial D} = f \quad \text{on } \partial D,$$

where the functions $f, g : \partial D \rightarrow \mathbb{R}$ are defined by

$$(6) \quad f(x) := \frac{1}{z(x)} \sum_{l=1}^N U_l [E_l], \quad g(x) := \frac{1}{z(x)} \sum_{l=1}^N [E_l]$$

and the contact impedance $z : \partial D \rightarrow \mathbb{R}$ is assumed to be a piecewise continuous function, with interfaces that are of zero surface measure, satisfying

$$0 < c_0 \leq z \leq c_1 \quad \text{a.e. on } \partial D.$$

For a given voltage pattern $U \in \mathbb{R}^N$ satisfying the grounding condition

$$(7) \quad \sum_{l=1}^N U_l = 0,$$

the equations (1) and (5) define the electric potential $u \in H^1(D)$ uniquely, cf. [57] and the variational form of the boundary value problem (1), (5) reads as follows: Given $U \in \mathbb{R}^N$ satisfying (7), find $u \in H^1(D)$ such that

$$(8) \quad \int_D \kappa \nabla u \cdot \nabla v \, dx + \langle g u|_{\partial D}, v|_{\partial D} \rangle_{\partial D} = \langle f, v|_{\partial D} \rangle_{\partial D} \quad \text{for all } v \in H^1(D).$$

Knowledge of u yields the corresponding electrode current vector $J \in \mathbb{R}^N$ via

$$(9) \quad J_l = \int_{E_l} \partial_{\kappa \nu} u \, d\sigma(x), \quad 1 \leq l \leq N.$$

3.2. A stochastic forward problem. The basic geometric setting of the stochastic problem we are interested in is as follows: Assume that the model domain is given by the lower hemisphere

$$D := B(0, R) \cap \mathbb{R}_-^d, \quad R > 0$$

and that ∂D is partitioned into two disjoint parts, namely the *accessible boundary* $\partial_1 D := \partial D \cap \mathbb{R}^{d-1}$ and the *inaccessible boundary* $\partial_2 D := \partial D \setminus \partial_1 D$, respectively. Such

a setting is found for instance in geophysical applications, where measurements can only be taken on the surface, cf., e.g., [55].

Let $(\Gamma, \mathcal{G}, \mathcal{P})$ be a probability space and let $\Theta : \Gamma \rightarrow \Gamma$ denote an ergodic d -dimensional *dynamical system*, i.e., a family of automorphisms $\{\Theta_x, x \in \mathbb{R}^d\}$ which satisfies the following conditions:

- (i) The family $\{\Theta_x, x \in \mathbb{R}^d\}$ is a group, i.e., $\Theta_0 = \text{id}$ and

$$\Theta_{x+y} = \Theta_x \Theta_y \quad \text{for all } x, y \in \mathbb{R}^d;$$

- (ii) the mappings $\Theta_x : \Gamma \rightarrow \Gamma, x \in \mathbb{R}^d$, preserve the measure \mathcal{P} on Γ , i.e., for every $B \in \mathcal{G}$, $\Theta_x B$ is \mathcal{P} -measurable and

$$\mathcal{P}(\Theta_x B) = \mathcal{P}(B);$$

- (iii) for every measurable function ϕ on $(\Gamma, \mathcal{G}, \mathcal{P})$, the function $(x, \omega) \mapsto \phi(\Theta_x \omega)$ is a measurable function on $(\mathbb{R}^d \times \Gamma, \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}, m \times \mathcal{P})$, where $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$ denotes the sigma-algebra generated by the measurable rectangles;

- (iv) the family $\{\Theta_x, x \in \mathbb{R}^d\}$ is ergodic, i.e., $\phi(\Theta_x \omega) = \phi(\omega)$ for all $x \in \mathbb{R}^d$ and \mathcal{P} -a.e. $\omega \in \Gamma$ implies $\phi = \text{const}$ \mathcal{P} -a.e.

Throughout this work we assume that the conductivity random field

$$\{\kappa(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$$

is the *stationary extension* on $\mathbb{R}^d \times \Gamma$ of some function $\kappa \in L^2(\Gamma; \mathbb{R}^{d \times d})$, that is,

$$(10) \quad (x, \omega) \mapsto \kappa(x, \omega) = \kappa(\Theta_x \omega).$$

Note that if κ can be written in the form (10) with a dynamical system $\{\Theta_x, x \in \mathbb{R}^d\}$ which satisfies (i)-(iii), then it is automatically *stationary* with respect to \mathcal{P} , i.e., for every finite collection of points $x^{(i)}, i = 1, \dots, k$, and any $h \in \mathbb{R}^d$ the joint distribution of

$$\kappa(x^{(1)} + h, \omega), \dots, \kappa(x^{(k)} + h, \omega)$$

under \mathcal{P} is the same as that of

$$\kappa(x^{(1)}, \omega), \dots, \kappa(x^{(k)}, \omega).$$

Even if it is not explicitly stated, we always assume that the conductivity random field may be written in the form (10), where the underlying dynamical system $\{\Theta_x, x \in \mathbb{R}^d\}$ satisfies conditions (i)-(iv). Moreover, we will explicitly state if the conductivity random field $\{\kappa(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ satisfies one of the following assumptions:

- (A1) $\kappa \in L^2(\Gamma; L^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d}))$ and the random field is strictly positive and uniformly bounded, that is, there exists a constant $c > 0$ such that for every $\xi \in \mathbb{R}^d$ and a.e. $x \in \mathbb{R}^d$

$$\mathcal{P}\left\{\omega \in \Gamma : c^{-1}|\xi|^2 \leq \xi \cdot \kappa(x, \omega)\xi \leq c|\xi|^2\right\} = 1.$$

- (A2) $\{\kappa(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ satisfies the *spectral gap property*, cf. [29]: There exist constants $\rho > 0$ and $r < \infty$ such that for all measurable functions on $\left\{\kappa : \mathbb{R}^d \rightarrow \{\kappa_0 \in \mathbb{R}^{d \times d} : |\kappa_0 \xi| \leq |\xi|, c|\xi|^2 \leq \xi \cdot \kappa_0 \xi \text{ for all } \xi \in \mathbb{R}^d\}\right\}$

$$\mathbb{V}\phi \leq \frac{1}{\rho} \mathbb{M} \int_{\mathbb{R}^d} \left(\text{osc}_{\kappa|_{B(x, r)}} \phi \right)^2 dx,$$

where we have set

$$\begin{aligned} \left(\text{osc}_{\kappa|_{B(x,r)}} \phi \right) &:= \sup \left\{ \phi(\tilde{\kappa}) : \tilde{\kappa} \in \Omega, \tilde{\kappa}|_{\mathbb{R}^d \setminus B(x,r)} = \kappa|_{\mathbb{R}^d \setminus B(x,r)} \right\} \\ &\quad - \inf \left\{ \phi(\tilde{\kappa}) : \tilde{\kappa} \in \Omega, \tilde{\kappa}|_{\mathbb{R}^d \setminus B(x,r)} = \kappa|_{\mathbb{R}^d \setminus B(x,r)} \right\}. \end{aligned}$$

To account for the highly heterogeneous properties of the background medium, the latter is modeled using the conductivity random field with appropriate scaling by a small parameter $\varepsilon > 0$, i.e.,

$$\kappa_\varepsilon(\cdot, \cdot) : \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}^{d \times d}, \quad \kappa_\varepsilon(x, \omega) := \kappa(x/\varepsilon, \omega).$$

If the correlation length of the conductivity random field κ is, say 1, then the correlation length of the scaled version κ_ε is of order ε and for $\varepsilon \ll 1$ we obtain thus a rapidly oscillating random field.

3.3. Stochastic forward and inverse problem. Now let us introduce a stochastic forward model based on the complete electrode model: We search for a random field $\{u_\varepsilon(x, \omega), (x, \omega) \in \overline{D} \times \Gamma\}$ for the electrical potential with $u_\varepsilon \in L^2(\Gamma; H_0^1(D \cup \partial_1 D))$ such that the stochastic conductivity equation

$$(11) \quad \nabla \cdot (\kappa_\varepsilon \nabla u_\varepsilon) = 0 \quad \text{in } D \times \Gamma$$

subject to the boundary conditions

$$(12) \quad \begin{aligned} \kappa_\varepsilon \nu \cdot \nabla u_\varepsilon|_{\partial_1 D} + g u_\varepsilon|_{\partial_1 D} &= f && \text{on } \partial_1 D \times \Gamma \\ u_\varepsilon|_{\partial_2 D} &= 0 && \text{on } \partial_2 D \times \Gamma \end{aligned}$$

is satisfied \mathcal{P} -a.s. The variational formulation of the forward problem is to find $u_\varepsilon \in L^2(\Gamma; H_0^1(D \cup \partial_1 D))$ such that

$$(13) \quad \mathbb{M} \left\{ \int_D \kappa_\varepsilon \nabla u_\varepsilon \cdot \nabla v \, dx + \langle g u_\varepsilon, v \rangle_{\partial_1 D} \right\} = \mathbb{M} \langle f, v \rangle_{\partial_1 D}$$

for all $v \in L^2(\Gamma; H_0^1(D \cup \partial_1 D))$. For a given voltage pattern $U \in \mathbb{R}^N$ the corresponding measurement data is given by the random current pattern $J(\varepsilon, \omega) = (J_1(\varepsilon, \omega), \dots, J_N(\varepsilon, \omega))^T$, defined for \mathcal{P} -a.e. $\omega \in \Gamma$ by

$$(14) \quad J_l(\varepsilon, \omega) = \frac{1}{|E_l|} \int_{E_l} \kappa_\varepsilon(x, \omega) \nu \cdot \nabla u_\varepsilon(x, \omega)|_{\partial_1 D} \, d\sigma(x), \quad l = 1, \dots, N.$$

Due to the assumption (A1), the well-posedness of the variational formulation follows from a straightforward application of the Lax-Milgram theorem. Moreover, standard arguments from measure theory show that the solution to the stochastic forward problem (11), (12) also solves (13), cf. [3].

4. REFLECTING DIFFUSION PROCESSES

In his seminal paper [23], Fukushima established a one-to-one correspondence between regular symmetric Dirichlet forms and symmetric Hunt processes, which is the foundation for the construction of stochastic processes via Dirichlet form techniques. Therefore we assume that the reader is familiar with the theory of symmetric Dirichlet forms, as elaborated for instance in the monograph [25].

Let us consider the following symmetric bilinear forms on $L^2(D)$:

$$(15) \quad \mathcal{E}(v, w) := \int_D \kappa \nabla v(x) \cdot \nabla w(x) \, dx, \quad v, w \in \mathcal{D}(\mathcal{E}) := H^1(D)$$

and for the particular case $\kappa \equiv 1/2$, which is of special importance, we set

$$(16) \quad \mathcal{E}^{\text{BM}}(v, w) := \frac{1}{2} \int_D \nabla v(x) \cdot \nabla w(x) dx, \quad v, w \in \mathcal{D}(\mathcal{E}^{\text{BM}}) := H^1(D).$$

The pair $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ defined by (15) is a strongly local, regular symmetric Dirichlet form on $L^2(D)$. In particular, there exist an \mathcal{E} -exceptional set $\mathcal{N} \subset \overline{D}$ and a conservative diffusion process $X = (\Omega, \mathcal{F}, \{X_t, t \geq 0\}, \mathbb{P}_x)$, starting from $x \in \overline{D} \setminus \mathcal{N}$ such that X is associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. Without loss of generality let us assume that X is defined on the *canonical sample space* $\Omega = C([0, \infty); \overline{D})$. It is well-known that the symmetric Hunt process associated with (16) is the reflecting Brownian motion. Therefore, we call the symmetric Hunt process associated with (15) a *reflecting diffusion process*.

Let us briefly recall the concept of the *boundary local time* of reflecting diffusion processes, see, e.g., [31, 49, 10], which will be crucial for the subsequent derivation of the Feynman-Kac formulae. If the diffusion process is the solution to a stochastic differential equation, say the reflecting Brownian motion, then the boundary local time is given by the one-dimensional process L in the Skorohod decomposition, which prevents the sample paths from leaving \overline{D} , i.e.,

$$(17) \quad X_t = x + W_t - \frac{1}{2} \int_0^t \nu(X_s) dL_s,$$

\mathbb{P}_x -a.s. for q.e. $x \in \overline{D}$. This boundary local time is a continuous non-decreasing process which increases only when $X_t \in \partial D$, namely for all $t \geq 0$ and q.e. $x \in \overline{D}$

$$L_t = \int_0^t [\partial D](X_s) dL_s,$$

\mathbb{P}_x -a.s. and

$$\mathbb{E}_x \int_0^t [\partial D](X_s) ds = 0.$$

Although the reflecting diffusion process associated with (15) does in general not admit a Skorohod decomposition of the form (17), we may still define a continuous one-dimensional process with these properties. More precisely, by the Lipschitz property of ∂D , we have that $D \cap B(x, r_D) = \{(\tilde{x}, x_d) : x_d > \gamma(\tilde{x})\} \cap B(x, r_D)$ and the Lipschitz function γ is differentiable a.e. with a bounded gradient. In particular, we have for every Borel set $B \subset \partial D \cap B(x, r_D)$ that

$$\sigma(B) = \int_{\{\tilde{x} : (\tilde{x}, \gamma(\tilde{x})) \in B\}} \left(1 + |\nabla \gamma(\tilde{x})|^2\right)^{1/2} d\tilde{x}$$

and a straightforward computation yields that the Lebesgue surface measure σ is a smooth measure with respect to $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ having finite energy, i.e.,

$$\int_{\partial D} |v| d\sigma(x) \leq c \|v\|_{\mathcal{E}_1} \quad \text{for all } v \in \mathcal{D}(\mathcal{E}) \cap C(\overline{D}),$$

where we have used the inner product $\mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + \langle \cdot, \cdot \rangle$.

Definition 4.1. The positive continuous additive functional of X whose Revuz measure is given by the Lebesgue surface measure σ on ∂D , i.e., the unique $L \in \mathcal{A}_c^+$ such that

$$(18) \quad \lim_{t \rightarrow 0^+} \frac{1}{t} \int_D \mathbb{E}_x \left\{ \int_0^t \phi(X_s) dL_s \right\} \psi(x) dx = \int_{\partial D} \phi(x) \psi(x) d\sigma(x)$$

for all non-negative Borel functions ϕ and all α -excessive functions ψ , is called the *boundary local time* of the reflecting diffusion process X .

The rest of this section is devoted to showing that the \mathcal{E} -exceptional set \mathcal{N} is actually empty. Therefore, we consider the non-positive definite self-adjoint operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ associated with the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$. That is, for $v \in \mathcal{D}(\mathcal{L})$ we have

$$(19) \quad \langle -\mathcal{L}v, w \rangle = \mathcal{E}(v, w) \quad \text{for all } w \in \mathcal{D}(\mathcal{E})$$

and the domain of \mathcal{L} is given by

$$\mathcal{D}(\mathcal{L}) = \left\{ v \in \mathcal{D}(\mathcal{E}) : \exists \phi \in L^2(D) \text{ s.t. } \mathcal{E}(v, w) = \int_D \phi w \, dx \, \forall w \in \mathcal{D}(\mathcal{E}) \right\}.$$

In order to *refine* the reflecting diffusion process X to start from every $x \in \overline{D}$, we exploit the connection between the strongly continuous sub-Markovian contraction semigroup $\{T_t, t \geq 0\}$ on $L^2(D)$ and the evolution system corresponding to $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$, see, e.g., the monograph [50]. Namely, for every $v_0 \in L^2(D)$, the trajectory $v : (0, T) \rightarrow H^1(D)$, $v(t) = T_t v_0$ belongs to the function space

$$\{\phi \in L^2((0, T); H^1(D)) : \dot{\phi} \in L^2((0, T); H^{-1}(D))\}$$

and is the unique mild solution to the parabolic *abstract Cauchy problem*

$$(20) \quad \begin{aligned} \dot{v} + \mathcal{L}v &= 0 \quad \text{in } (0, T) \\ v(0) &= v_0. \end{aligned}$$

This is equivalent to the variational formulation

$$(21) \quad - \int_0^T \langle v(t), w \rangle \dot{\varphi}(t) \, dt + \int_0^T \langle \mathcal{L}v(t), w \rangle \varphi(t) \, dt - \langle v_0, w \rangle \varphi(0) = 0$$

for all $w \in H^1(D)$ and all $\varphi \in C_c^\infty([0, T])$. Moreover, T_t is known to be a bounded operator from $L^1(D)$ to $L^\infty(D)$ for every $t > 0$. Therefore, by the Dunford-Pettis theorem, it can be represented as an integral operator for every $t > 0$,

$$(22) \quad T_t \phi(x) = \int_D p(t, x, y) \phi(y) \, dy \quad \text{for every } \phi \in L^1(D),$$

where for all $t > 0$ we have $p(t, \cdot, \cdot) \in L^\infty(D \times D)$ and $p(t, \cdot, \cdot) \geq 0$ a.e. We call the function p the *transition kernel density* of X .

The following proposition adapts a well-known result for diffusion processes on \mathbb{R}^d , cf. [58], which follows from the famous De Giorgi-Nash-Moser theorem, to the case of reflecting diffusion processes on \overline{D} . The key idea of the proof is the following *extension by reflection* technique from [62, Section 2.4.3]: We extend the solution to a parabolic problem by reflection at the boundary. Then we show that this extension again solves a parabolic problem so that we can apply the interior regularity result due to De Giorgi, Nash and Moser. See also the article [46] by Nittka, where such a technique is applied to elliptic boundary value problems.

Proposition 1. *$p \in C^{0,\delta}((0, T] \times \overline{D} \times \overline{D})$ for some $\delta \in (0, 1)$, i.e., for each fixed $0 < t_0 \leq T$, there exists a positive constant c such that*

$$(23) \quad |p(t_2, x_2, y_2) - p(t_1, x_1, y_1)| \leq c(\sqrt{t_2 - t_1} + |x_2 - x_1| + |y_2 - y_1|)^\delta$$

for all $t_0 \leq t_1 \leq t_2 \leq T$ and all $(x_1, y_1), (x_2, y_2) \in \overline{D} \times \overline{D}$. Moreover, the mapping $t \mapsto p(t, \cdot, \cdot)$ is analytic from $(0, \infty)$ to $C^{0,\delta}(\overline{D} \times \overline{D})$.

Proof. First note that Nash's inequality holds for the underlying Dirichlet form $(\mathcal{E}, H^1(D))$, i.e., there exists a constant $c_1 > 0$ such that

$$\|v\|_2^{2+4/d} \leq c_1(\mathcal{E}(v, v) + \|v\|_2^2)\|v\|_1^{4/d} \quad \text{for all } v \in H^1(D).$$

This is a direct consequence of the uniform ellipticity (2) and [5, Corollary 2.2], where Nash's inequality is shown to hold for the Dirichlet form $(\mathcal{E}^{\text{BM}}, H^1(D))$ for a bounded Lipschitz domain D . Analogously to the proof of [5, Theorem 3.1], it follows thus from [12, Theorem 3.25] that the transition kernel density satisfies an Aronson type Gaussian upper bound

$$(24) \quad p(t, x, y) \leq c_1 t^{-d/2} \exp\left(-\frac{|x-y|^2}{c_2 t}\right)$$

for all $t \leq 1$ and all $(x, y) \in \overline{D} \times \overline{D}$. In particular, $\sup_{0 < t \leq 1} \|p(t, \cdot, \cdot)\|_\infty$ is finite and hence by the interior Hölder continuity obtained from the De Giorgi-Nash-Moser theorem, cf. [45, 58], the estimate (23) is true for all $(x_1, y_1), (x_2, y_2)$ satisfying $d(x_i, \partial D), d(y_i, \partial D) > c_3$, $i = 1, 2$, for some constant $c_3 > 0$ and all $t_0 \leq t_1 \leq t_2 \leq 1$. Note that by the semigroup property the Chapman-Kolmogorov equation holds, i.e.,

$$(25) \quad p(t_1 + t_2, x, y) = \int_D p(t_1, x, z) p(t_2, z, y) dz$$

for every pair $t_1, t_2 \geq 0$ and a.e. $x, y \in \overline{D}$. In particular, for fixed $y \in \overline{D}$ the function $v := p(\cdot, \cdot, y)$ is the unique solution to (20) with initial value $v_0 := p(0, \cdot, y) \in L^2(D)$. Now let $z \in \partial D$ so that by the Lipschitz property of ∂D we have after translation and rotation $B(z, r_D) \cap \overline{D} = \{(\tilde{x}, x_d) \in B(z, r_D) : x_d \geq \gamma(\tilde{x})\}$ and $B(z, r_D) \cap \partial D = \{\tilde{x} \in B(z, r_D) : x_d = \gamma(\tilde{x})\}$, where we have introduced the notation $\tilde{x} = (x_1, \dots, x_{d-1})^T$. Let us furthermore introduce the one-to-one transformation $\Psi(x) := (\tilde{x}, x_d - \gamma(\tilde{x}))$ which straightens the boundary $B(z, r_D) \cap \partial D$. Ψ is a bi-Lipschitz transformation and the Jacobians of both Ψ and Ψ^{-1} are bounded with bounds that depend only on the Lipschitz constant c_D . Since v is the solution to (20) with appropriate initial condition, the function $\hat{v} := v(\cdot, \Psi^{-1}(\cdot))$ must satisfy the variational formulation of the following parabolic problem in $\hat{D}(z, r_D) := \Psi(B(z, r_D) \cap \overline{D})$

$$\begin{aligned} \int_0^T \dot{\varphi}(t) \int_{\hat{D}(z, r_D)} \hat{v}(t) w dx dt &= - \sum_{i,j=1}^d \int_0^T \varphi(t) \int_{\hat{D}(z, r_D)} \hat{\kappa}_{ij} \partial_i \hat{v}(t) \partial_j w dx dt \\ &\quad - \varphi(0) \int_{\hat{D}(z, r_D)} \hat{v}_0 w dx \end{aligned}$$

for all $w \in C_c^\infty(\hat{D}(z, r_D))$ and all $\varphi \in C_c^\infty([0, T])$. The coefficient $\hat{\kappa}$ is obtained via change of variables and it is bounded and uniformly elliptic by the boundedness of the Jacobians of Ψ and Ψ^{-1} , respectively. Now we use reflection at the hyperplane $\{(\tilde{y}, 0)\}$ via the mapping $\rho(x) := (\tilde{x}, -x_d)$ which yields that the function $\hat{v}(\cdot, \rho(\cdot))$ satisfies the variational formulation of a parabolic problem on $\rho(\hat{D}(z, r_D))$. Summing up both variational formulations on $\hat{D}(z, r_D)$ and on $\rho(\hat{D}(z, r_D))$, respectively, we obtain that the function

$$\tilde{v}(t, x) := \begin{cases} \hat{v}(t, x), & x \in \hat{D}(z, r_D) \\ \hat{v}(t, \rho(x)), & x \in \rho(\hat{D}(z, r_D)) \end{cases}$$

satisfies the variational formulation of a parabolic problem in $\hat{D}(z, r_D) \cup \rho(\hat{D}(z, r_D))$. By the interior Hölder estimate for \tilde{v} , together with the fact that we may choose

$c_3 = r_D/4c_D$, we obtain thus

$$|p(t_2, x_2, \Psi^{-1}(y_2)) - p(t_1, x_1, \Psi^{-1}(y_1))| \leq c_1(\sqrt{t_2 - t_1} + |y_2 - y_1|)^{c_2}$$

for all $t_0 \leq t_1 \leq t_2 \leq 1$ and $y_1, y_2 \in \{(\tilde{x}, x_d) : |\tilde{x}| < c_3, x_d \in (0, r_D/4)\}$. As Ψ is bi-Lipschitz, for fixed x , the mapping $(t, y) \mapsto p(t, x, y)$ is Hölder continuous in $(t_0, 1] \times (B(z, c_3) \cap \overline{D})$ and by symmetry of the transition kernel density the same holds true for the mapping $(t, x) \mapsto p(t, x, y)$ for fixed y . Finally, the first assertion on $(t_0, 1] \times \overline{D} \times \overline{D}$ follows due to compactness of ∂D and its generalization to arbitrary $T > 0$ is obtained after repeatedly applying the Chapman-Kolmogorov equation.

The second assertion follows by the fact that the semigroup $\{T_t, t \geq 0\}$ extrapolates to a holomorphic semigroup on $L^2(D)$. More precisely, the semigroup possesses a holomorphic extension to the sector $\Sigma_\theta := \{re^{i\alpha} : r > 0, |\alpha| < \theta\}$ for some $\theta \in (0, \frac{\pi}{2}]$, cf., e.g., [50]. Let $0 < t_0 \leq T$ and set

$$\Sigma_\theta(t_0, T) := \{z \in \mathbb{C} : z - t_0 \in \Sigma_\theta, |z| < T\}.$$

By the Hölder continuity of p , the set $\{p(z, \cdot, \cdot) : z \in \Sigma_\theta(t_0, T)\}$ is a bounded subset of $C^{0,\delta}(\overline{D} \times \overline{D})$. Moreover, the family of functionals obtained from integration against the functions $[B_1](x)[B_2](y)$ for measurable $B_1, B_2 \subset \overline{D}$ form a separating subspace of $(C^{0,\delta}(\overline{D}, \overline{D}))'$, i.e., for $k \in C^{0,\delta}(\overline{D} \times \overline{D})$

$$\int_{D \times D} k(x, y)[B_1](x)[B_2](y) dx dy = 0 \quad \text{for all measurable } B_1, B_2 \subset \overline{D}$$

implies that $k \equiv 0$. As the mapping

$$z \mapsto \langle T_z[B_1], [B_2] \rangle = \int_{D \times D} p(z, x, y)[B_1](y)[B_2](x) dx dy$$

is holomorphic for all $z \in \Sigma_\theta$, the mapping $z \mapsto p(z, \cdot, \cdot)$ is holomorphic from $\Sigma_\theta(t_0, T)$ to $C^{0,\delta}(\overline{D} \times \overline{D})$ by [1, Theorem 3.1]. Since t_0 and T were arbitrary, the assertion is proved. \square

By [24, Theorem 2], the existence of a Hölder continuous transition kernel density ensures that we may refine the process X to start from every $x \in \overline{D}$ by identifying the strongly continuous semigroup $\{T_t, t \geq 0\}$ with the transition semigroup $\{P_t, t \geq 0\}$. In particular, if v is continuous and locally in $H^1(D)$, the Fukushima decomposition holds for every $x \in \overline{D}$, i.e.,

$$(26) \quad v(X_t) = v(X_0) + M_t^v + N_t^v, \quad \text{for all } t > 0,$$

\mathbb{P}_x -a.s., where M^v is a martingale additive functional of X having finite energy and N^v is a continuous additive functional of X having zero energy.

Moreover, both M^v and N^v can be taken to be additive functionals of X in the strict sense, cf. [25, Theorem 5.2.5].

Finally, note that the 1-potential of the Lebesgue surface measure σ of ∂D is the solution to an elliptic boundary value problem on a Lipschitz domain with bounded data. By elliptic regularity theory, cf., e.g., [30], this solution is continuous, implying that the boundary local time L exists as a positive continuous additive functional in the strict sense, cf. [25, Theorem 5.1.6].

5. SKOROHOD DECOMPOSITIONS

In this section, we derive Skorohod decompositions of the reflecting diffusion process X for two practically relevant special cases, namely local Lipschitz conductivities and isotropic piecewise constant conductivities.

The assertion of the following proposition is already covered by [26, Theorem 2.3]; we include a proof for the sake of self-containedness.

Proposition 2. *Let $\kappa \in C_{loc}^{0,1}(\overline{D}; \mathbb{R}^{d \times d})$ be a symmetric, uniformly bounded and uniformly elliptic conductivity. Then the reflecting diffusion process X admits the following Skorohod decomposition*

$$(27) \quad X_t = x + \int_0^t B(X_s) dW_s + \int_0^t \nabla \kappa(X_s) ds - \int_0^t \kappa(X_s) \nu(X_s) dL_s,$$

\mathbb{P}_x -a.s., where $B : \overline{D} \rightarrow \mathbb{R}^{d \times d}$ denotes the positive definite diffusion matrix satisfying $B^2 = 2\kappa$, W is a standard d -dimensional Brownian motion and L is the boundary local time of X .

Proof. We have shown in Section 2, that the Fukushima decomposition holds with a unique martingale additive functional M^v in the strict sense and a unique continuous additive functional N^v in the strict sense. Let us first compute the energy measure of M^v . For $v, w \in \mathcal{D}(\mathcal{E})$ we obtain

$$\begin{aligned} \int_D w(x) d\mu_{\langle M^v \rangle}(x) &= \lim_{t \rightarrow 0+} \frac{1}{t} \int_D \mathbb{E}_x \{ (v(X_t) - v(x_0))^2 \} w(x) dx \\ &= \lim_{t \rightarrow 0+} \frac{1}{t} \int_D (T_t v^2(x) - 2v(x)T_t v(x) + v^2(x)) w(x) dx \\ &= \lim_{t \rightarrow 0+} \frac{2}{t} \int_D v(x) w(x) (v(x) - T_t v(x)) dx \\ &\quad - \lim_{t \rightarrow 0+} \frac{1}{t} \int_D v^2(x) (w(x) - T_t w(x)) dx \\ &= 2\mathcal{E}(vw, v) - \mathcal{E}(v^2, w) \\ &= 2 \int_D \kappa \nabla v(x) \cdot \nabla v(x) w(x) dx, \end{aligned}$$

which yields the energy measure

$$d\mu_{\langle M^v \rangle}(x) = 2 \sum_{i,j=1}^d \kappa_{ij}(x) \partial_i v(x) \partial_j v(x) dx$$

so that the predictable quadratic variation of M^v is given by

$$(28) \quad \langle M^v \rangle_t = 2 \int_0^t \sum_{i,j=1}^d \kappa_{ij}(X_s) \partial_i v(X_s) \partial_j v(X_s) ds.$$

Using the coordinate mappings $\phi_i(x) := x_i$, $i = 1, \dots, d$, on \overline{D} yields that M^ϕ is a continuous martingale additive functional in the strict sense with covariation

$$\langle M^{\phi_i}, M^{\phi_j} \rangle_t = 2 \int_0^t \kappa_{ij}(X_s) ds,$$

\mathbb{P}_x -a.s. A standard characterization of continuous martingales, cf., e.g., [?], yields that

$$(29) \quad M_t^v = \int_0^t (B(X_s) \nabla v(X_s)) \cdot dW_s,$$

\mathbb{P}_x -a.s., where $B : \overline{D} \rightarrow \mathbb{R}^{d \times d}$ denotes the positive definite diffusion matrix satisfying $B^2 = 2\kappa$ and W is a standard d -dimensional Brownian motion.

Now let us consider the continuous additive functional N^v . Again using the coordinate mappings on \overline{D} , we obtain from Green's formula that

$$\begin{aligned} \mathcal{E}(\phi_i, w) &= \sum_{j=1}^d \int_D \kappa_{ij}(x) \partial_j w(x) dx \\ &= - \sum_{j=1}^d \int_D \partial_j \kappa_{ij}(x) w(x) dx + \sum_{j=1}^d \int_{\partial D} \kappa_{ij}(x) \nu_j(x) w(x) d\sigma(x) \end{aligned}$$

for all $v \in H^1(D) \cap C(\overline{D})$. That is, N^{ϕ_i} is associated with the signed Radon measure

$$- \sum_{j=1}^d \partial_j \kappa_{ij} dx + \sum_{j=1}^d \kappa_{ij}(x) \nu_j(x) d\sigma(x)$$

and by the fact that the unique positive continuous additive functionals in the strict sense having as Revuz measure the Lebesgue measure and the Lebesgue surface measure, respectively, are given by the constant additive functional t and the boundary local time L_t , we have shown that for every $x \in \overline{D}$

$$(30) \quad N_t^{\phi_i} = \sum_{j=1}^d \int_0^t \partial_j \kappa_{ij}(X_s) ds - \sum_{j=1}^d \int_0^t \kappa_{ij}(X_s) \nu_j(X_s) dL_s,$$

\mathbb{P}_x -a.s. Substitution of (29) and (30) in the Fukushima decomposition for the coordinate mappings finally yields the Skorohod decomposition (32). \square

Now let us turn to the case of isotropic piecewise constant conductivities and for simplicity of the presentation let us consider a simplistic two-phase medium, where

$$(31) \quad \kappa(x) = \begin{cases} \kappa_1, & x \in D_1 \\ \kappa_2, & x \in D_2, \end{cases}$$

with constants $\kappa_1, \kappa_2 > 0$ and D is a simply connected bounded Lipschitz domain which consists of two disjoint subdomains such that $D_1 = D \setminus \overline{D}_2$. We assume that D_2 is a simply connected Lipschitz domain. ν is the outer unit normal vector on ∂D and the outer unit normal vector on ∂D_2 with respect to D_2 .

Definition 5.1. The positive continuous additive functional L^0 of X whose Revuz measure is given by the scaled Lebesgue surface measure $(\kappa_1 + \kappa_2)\sigma$ on ∂D_2 is called the *symmetric local time* of the reflecting diffusion process X at ∂D_2 .

Remark 1. The term “symmetric” comes from the fact that in the one-dimensional case L^0 is the local time defined by the Tanaka formula with the convention $\text{sign}(0) = 0$, which is called the symmetric local time, see [?]. In this case we have

$$L_t^0 = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^t [[-\varepsilon, \varepsilon]](X_s) ds.$$

Proposition 3. *Let κ be given by (31). Then the reflecting diffusion process X admits the following Skorohod decomposition*

$$(32) \quad X_t = x + \int_0^t \sqrt{2\kappa(X_s)} dW_s + \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \int_0^t \nu(X_s) dL_s^0 - \kappa_1 \int_0^t \nu(X_s) dL_s,$$

\mathbb{P}_x -a.s., where W is a standard d -dimensional Brownian motion, L^0 is the symmetric local time of X at ∂D_2 and L is the boundary local time.

Proof. Repeating the computations from the proof of Proposition 2 yields for the martingale additive functional the predictable quadratic variation

$$\langle M^v \rangle_t = 2 \int_0^t \kappa(X_s) |\nabla v(X_s)|^2 ds, \quad \mathbb{P}_x\text{-a.s. for every } x \in \overline{D},$$

implying that

$$M_t^v = \int_0^t \sqrt{2\kappa(X_s)} \nabla v(X_s) \cdot dW_s, \quad \mathbb{P}_x\text{-a.s. for every } x \in \overline{D},$$

where W is a standard d -dimensional Brownian motion. By Green's formula we have for all $w \in \mathcal{D}(\mathcal{E}) \cap C(\overline{D})$

$$\begin{aligned} \mathcal{E}(v, w) &= \kappa_1 \int_{D_1} \nabla v \cdot \nabla w dx + \kappa_2 \int_{D_2} \nabla v \cdot \nabla w dx \\ &= - \int_D v \kappa \Delta v dx - (\kappa_1 - \kappa_2) \int_{\partial D_2} \partial_\nu v w d\sigma(x) \\ &\quad + \kappa_1 \int_{\partial D} \partial_\nu v w d\sigma(x). \end{aligned}$$

Using the coordinate mappings on \overline{D} we obtain that N^{ϕ_i} is associated with the signed Radon measure

$$-(\kappa_1 - \kappa_2) \nu(x) d\sigma|_{\partial D_2}(x) + \kappa_1 \nu(x) d\sigma|_{\partial D}(x).$$

The assertion follows by the fact that the unique positive continuous additive functionals in the strict sense having as Revuz measure the (scaled) Lebesgue surface measure on ∂D_2 and ∂D , respectively, are given by the symmetric local time L^0 at ∂D_2 and the boundary local time L_t . \square

6. FEYNMAN-KAC FORMULAE

In this section, we derive the Feynman-Kac formulae for both the continuum model and the complete electrode model. Afterwards, we will obtain, as a corollary, a Feynman-Kac formula for the mixed boundary value problem corresponding to the stochastic anomaly detection problem introduced in the first chapter. Compared to the earlier works [31, 49, 10] on Feynman-Kac formulae, the main difficulty in deriving these formulae in our particular setting comes from the lack of Itô's formula for general reflecting diffusion processes.

The rest of this subsection is devoted to providing some auxiliary lemmata.

Lemma 6.1. *The transition kernel density p approaches the stationary distribution uniformly and exponentially fast, that is, there exist positive constants c_1 and c_2 such that for all $(x, y) \in \overline{D} \times \overline{D}$ and every $t \geq 0$,*

$$(33) \quad |p(t, x, y) - |D|^{-1}| \leq c_1 \exp(-c_2 t).$$

Proof. Fix $x \in \overline{D}$. Then by the Chapman-Kolmogorov equation

$$\begin{aligned} p(t, x, x) - |D|^{-1} &= \int_D p(t/2, x, y) p(t/2, y, x) dy - |D|^{-1} \\ &= \int_D (p(t/2, x, y))^2 dy - |D|^{-1} \\ &= \int_D (p(t/2, x, y) - |D|^{-1})^2 dy \geq 0, \end{aligned}$$

where we have used that $\int_D p(t, x, y) dy = 1$ for all $t \geq 0$ and $x \in \overline{D}$. Moreover, we have by the analyticity of the mapping $t \mapsto p(t, x, x)$, cf. Theorem 1, and the fact that p solves a parabolic boundary value problem with homogeneous Neumann boundary condition

$$\begin{aligned} \frac{d}{dt}(p(t, x, x) - |D|^{-1}) &= \frac{d}{dt} \int_D p(t/2, x, y) p(t/2, y, x) dy \\ &= \frac{d}{dt} \int_D (p(t/2, x, y))^2 dy \\ &= - \int_D \kappa(y) \nabla_y p(t/2, x, y) \cdot \nabla_y p(t/2, x, y) dy \\ &\leq -c^{-1} \int_D \nabla_y p(t/2, x, y) \cdot \nabla_y p(t/2, x, y) dy \\ &\leq -c^{-1} c_D^{-1} \left(\int_D (p(t/2, x, y))^2 dy - |D|^{-1} \right) \\ &= -c^{-1} c_D^{-1} (p(t, x, x) - |D|^{-1}), \end{aligned}$$

where we have used (2) and the Poincaré inequality

$$\left\| \phi - |D|^{-1} \int_D \phi(x) dx \right\|_2 \leq c_D \|\nabla \phi\|_2 \quad \text{for all } \phi \in H^1(D).$$

Integration of the inequality from above yields a diagonal estimate, i.e., there exist positive constants c_1 and c_2 such that

$$0 \leq p(t, x, x) - |D|^{-1} \leq c_1 \exp(-c_2 t) \quad \text{for every } t \geq 0.$$

Now, the assertion follows from the Cauchy-Schwarz inequality and the fact that, by the computations from above, we may write the expression $|p(t, x, y) - |D|^{-1}|$ in the form

$$\begin{aligned} &\left| \int_D (p(t/2, x, z) - |D|^{-1})(p(t/2, z, y) - |D|^{-1}) dz \right| \\ &\leq \left(\int_D (p(t/2, x, z) - |D|^{-1})^2 dz \right)^{1/2} \left(\int_D (p(t/2, y, z) - |D|^{-1})^2 dz \right)^{1/2} \\ &\leq \left(p(t/2, x, x) - |D|^{-1} \right)^{1/2} \left(p(t/2, y, y) - |D|^{-1} \right)^{1/2} \end{aligned}$$

□

Lemma 6.2. *Let $\kappa_{ij} \in C^\infty(\overline{D}; \mathbb{R}^{d \times d})$, $1 \leq i, j \leq d$. Then the set*

$$(34) \quad V_\kappa(D) := \{\phi : \phi \in C^2(D), \partial_{\kappa\nu} \phi = 0 \text{ a.e. on } \partial D\} \cap H^1(D)$$

is dense in $H^1(D)$.

Proof. Diagonalizing the operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ corresponding to the conductivity κ , we obtain an orthonormal basis $\{\phi_k, k \in \mathbb{N}\}$ of $L^2(D)$ and an increasing sequence $(\lambda_k)_{k \in \mathbb{N}}$ of real positive numbers which tend to infinity such that for every $k \in \mathbb{N}$, $\phi_k \in H^1(D)$ is a weak solution of the corresponding eigenvalue problem with homogeneous Neumann boundary condition. Note that the inner product $\mathcal{E}_1(\cdot, \cdot)$ is equivalent to the standard inner product on $H^1(D)$. Note that $V_\kappa(D)$ contains the linear span of $\{\phi_k, k \in \mathbb{N}\}$ by elliptic regularity so it is enough to show that the linear span of eigenfunctions is dense. Therefore, let $\psi \in H^1(D)$ such that $\mathcal{E}_1(\phi_k, \psi) = 0$ for every $k \in \mathbb{N}$, then

$$0 = \int_D \kappa \nabla \phi_k \cdot \nabla \psi \, dx + \langle \phi_k, \psi \rangle = (\lambda_k + 1) \langle \phi_k, \psi \rangle.$$

Hence it follows $\langle \phi_k, \psi \rangle = 0$ for every $k \in \mathbb{N}$ and the fact that $\{\phi_k, k \in \mathbb{N}\}$ is an orthonormal basis of $L^2(D)$ implies $\psi \equiv 0$ which proves the assertion. \square

Lemma 6.3. *For every $x \in \overline{D}$ and every bounded Borel function ϕ on ∂D we have*

$$(35) \quad \mathbb{E}_x \int_0^t \phi(X_s) \, dL_s = \int_0^t \int_{\partial D} p(s, x, y) \phi(y) \, d\sigma(y) \, ds \quad \text{for all } t \geq 0.$$

Proof. First, the expression (35) is well-defined since the boundary local time of X exists as a positive continuous additive functional in the strict sense. Without loss of generality we may assume that ϕ is non-negative. It follows from [25, Theorem 5.1.3] that the Revuz correspondence (18) is equivalent to

$$\begin{aligned} \int_D \psi(x) \mathbb{E}_x \int_0^t \phi(X_s) \, dL_s \, dx &= \int_0^t \int_{\partial D} \phi(y) T_s \psi(y) \, d\sigma(y) \, ds \\ &= \int_D \psi(x) \int_0^t \int_{\partial D} p(s, y, x) \phi(y) \, d\sigma(y) \, ds \, dx \end{aligned}$$

for every $t > 0$ and all non-negative Borel functions ψ and ϕ , where we have used Fubini's theorem in the second line. As this holds for every non-negative Borel function ψ , we deduce

$$\mathbb{E}_x \int_0^t \phi(X_s) \, dL_s = \int_0^t \int_{\partial D} p(s, x, y) \phi(y) \, d\sigma(y) \, ds \quad \text{a.e. in } \overline{D}.$$

To obtain the assertion everywhere in \overline{D} , fix an arbitrary $x_0 \in \overline{D}$ and consider for $t_0 > 0$ the integral

$$\begin{aligned} \mathbb{E}_{x_0} \int_{t_0}^t \phi(X_s) \, dL_s &= \int_D p(t_0, x_0, x) \mathbb{E}_x \left\{ \int_0^{t-t_0} \phi(X_s) \, dL_s \right\} \, dx \\ &= \int_D p(t_0, x_0, x) \left(\int_0^{t-t_0} \int_{\partial D} p(s, x, y) \phi(y) \, d\sigma(y) \, ds \right) \, dx \\ &= \int_{t_0}^t \int_{\partial D} p(s, x_0, y) \phi(y) \, d\sigma(y) \, ds, \end{aligned}$$

where we have used the Markov property of X . Now let $(t_k)_{k \in \mathbb{N}}$ denote a positive sequence which monotonically decreases to zero as $k \rightarrow \infty$. By the computation from above we have for every $x \in \overline{D}$

$$\mathbb{E}_x \int_0^t \phi(X_s) \, dL_s = \int_{t_k}^t \int_{\partial D} p(s, x, y) \phi(y) \, d\sigma(y) \, ds + \mathbb{E}_x \int_0^{t_k} \phi(X_s) \, dL_s$$

The claim follows by the facts that ϕ is bounded and $\mathbb{E}_x L_{t_k}$ goes to zero as $k \rightarrow \infty$ which follows from monotonicity and continuity of the local time and the property $L_0 = 0$ \mathbb{P}_x -a.s. for every $x \in \overline{D}$. \square

6.1. Continuum model. The main result for the continuum model (1), (3) is the following theorem.

Theorem 6.4. *Let f be a bounded Borel function satisfying $\langle f, 1 \rangle_{\partial D} = 0$. Then there is a unique weak solution $u \in C(\overline{D}) \cap H_{\diamond}^1(D)$ to the boundary value problem (1), (3). This solution admits the Feynman-Kac representation*

$$(36) \quad u(x) = \lim_{t \rightarrow \infty} \mathbb{E}_x \int_0^t f(X_s) dL_s \quad \text{for all } x \in \overline{D}.$$

Proof. The existence of a unique normalized weak solution u to (1), (3) is guaranteed by the standard theory of linear elliptic boundary value problems. Let us set

$$u_t(x) := \mathbb{E}_x \int_0^t f(X_s) dL_s \quad \text{and} \quad u_{\infty}(x) := \lim_{t \rightarrow \infty} u_t(x), \quad x \in \overline{D},$$

respectively. From the occupation formula (35) and the compatibility condition (4), it follows immediately that

$$u_t(x) = \int_0^t \int_{\partial D} (p(s, x, y) - |D|^{-1}) f(y) d\sigma(y) ds \quad \text{for all } x \in \overline{D}.$$

By Lemma 6.1 the convergence towards the stationary distribution is uniform over \overline{D} , in particular,

$$(37) \quad u_{\infty}(x) = \int_0^{\infty} \int_{\partial D} (p(s, x, y) - |D|^{-1}) f(y) d\sigma(y) ds \quad \text{for all } x \in \overline{D}.$$

It follows from (37) together with the Hölder continuity shown in Proposition 1 and the Aronson type upper bound (24) that u_{∞} is in $C(\overline{D})$. Moreover, by the facts that p is the transition kernel density of a reflecting diffusion process and f is bounded, Fubini's theorem yields

$$\int_D \left(\int_0^{\infty} \int_{\partial D} p(s, x, y) f(y) d\sigma(y) ds \right) dx = \int_0^{\infty} \int_{\partial D} f(y) d\sigma(y) ds = 0,$$

i.e., $u_{\infty} \in H_{\diamond}^1(D)$.

Now, let us use the following regularization technique in order to show $u \equiv u_{\infty}$: Let $(\kappa^{(k)})_{k \in \mathbb{N}}$ denote a sequence of smooth conductivities with components in $C^{\infty}(\overline{D})$ such that for $1 \leq i, j \leq d$, $\kappa_{ij}^{(k)} \rightarrow \kappa_{ij}$ a.e. as $k \rightarrow \infty$. Let us consider the Dirichlet forms $(\mathcal{E}^{(k)}, H^1(D))$, $k \in \mathbb{N}$, with

$$\mathcal{E}^{(k)}(v, w) := \int_D \kappa^{(k)} \nabla v \cdot \nabla w dx$$

and the associated reflecting diffusion processes $X^{(k)}$. By Proposition 2, we obtain the Skorohod decomposition

$$X_t^{(k)} = x + \int_0^t a^{(k)}(X_s^{(k)}) ds + \int_0^t B^{(k)}(X_s) dW_s - \int_0^t \kappa^{(k)}(X_s^{(k)}) \nu(X_s^{(k)}) dL_s^{(k)},$$

where W is a standard d -dimensional Brownian motion, $a_i^{(k)} := \sum_{j=1}^d \partial_j \kappa_{ij}^{(k)}$, $i = 1, \dots, d$, and the matrix $B^{(k)}$ satisfies $2\kappa^{(k)} = (B^{(k)})^2$. Let us define $u_t^{(k)}$ in the same

manner as u_t and $u^{(k)}(x) := \lim_{t \rightarrow \infty} u_t^{(k)}(x)$, $x \in \overline{D}$. We show that $u^{(k)}$ is the unique weak solution of the elliptic boundary value problem

$$\begin{cases} \nabla \cdot (\kappa^{(k)} \nabla u^{(k)}) = 0 & \text{in } D \\ \partial_{\kappa^{(k)} \nu} u^{(k)} = f & \text{on } \partial D \end{cases}$$

in the Sobolev space $H_\diamond^1(D)$. For test functions $v \in V_{\kappa^{(k)}}(D)$, we may apply Itô's formula for semimartingales to obtain

$$\mathbb{E}_x v(X_t^{(k)}) = v(x) + \mathbb{E}_x \int_0^t \nabla \cdot (\kappa^{(k)} \nabla v(X_s^{(k)})) ds.$$

By Fubini's theorem, this is equivalent to

$$T_t^{(k)} v(x) - v(x) = \int_0^t \int_D p^{(k)}(s, x, y) \nabla \cdot (\kappa^{(k)} \nabla v(y)) dy ds,$$

where we have used the superscript “ (k) ” for the semigroup and transition kernel density, respectively, corresponding to $\kappa^{(k)}$. Multiplication with f , integration over ∂D and another change of the orders of integration finally yield

$$\int_{\partial D} f(y) (T_t^{(k)} v(y) - v(y)) d\sigma(y) = \langle u_t^{(k)}, \nabla \cdot (\kappa^{(k)} \nabla v) \rangle.$$

Since $u_t^{(k)} \rightarrow u^{(k)}$ and $T_t^{(k)} v \rightarrow |D|^{-1} \int_D v dx$, both uniformly on \overline{D} , as $t \rightarrow \infty$, we have

$$\langle u^{(k)}, \nabla \cdot (\kappa^{(k)} \nabla v) \rangle = - \langle f, v \rangle_{\partial D},$$

where we have used the expression (37) with $p^{(k)}$ instead of p for $u^{(k)}$. As this holds true for every $v \in V_{\kappa^{(k)}}(D)$, $u^{(k)}$ must be the unique normalized weak solution to the boundary value problem by a density argument.

Next, we show the convergence of the sequence $(u^{(k)})_{k \in \mathbb{N}}$ as $k \rightarrow \infty$ towards $u \in H_\diamond^1(D)$, the unique solution to (1), (3). From our assumptions on the sequence $(\kappa^{(k)})_{k \in \mathbb{N}}$, it is clear that $\kappa_{ij}^{(k)} \rightarrow \kappa_{ij}$, $1 \leq i, j \leq d$, in $L^2(D)$ as $k \rightarrow \infty$, which implies G -convergence of the sequence of elliptic operators $(\mathcal{L}^{(k)})_{k \in \mathbb{N}}$ on D towards \mathcal{L} , cf. [64]. That is, for any $\phi \in (H_0^1(D))'$, the solutions $w^{(k)} \in H_0^1(D)$ to the homogeneous Dirichlet problem

$$(38) \quad \nabla \cdot (\kappa^{(k)} \nabla w^{(k)}) = \phi \quad \text{in } D$$

satisfy $w^{(k)} \rightharpoonup w$ in $H_0^1(D)$ as $k \rightarrow \infty$ and $\kappa^{(k)} \nabla w^{(k)} \rightharpoonup \kappa \nabla w$ in $L^2(D; \mathbb{R}^d)$ as $k \rightarrow \infty$, where $w \in H_0^1(D)$ is the solution to the homogeneous Dirichlet problem

$$\nabla \cdot (\kappa \nabla w) = \phi \quad \text{in } D.$$

Consider the variational form of the Neumann problem for $u^{(k)} \in H_\diamond^1(D)$, i.e.,

$$\int_D \kappa^{(k)} \nabla u^{(k)} \cdot \nabla v dx = \langle f, v \rangle_{\partial D} \quad \text{for all } v \in H_\diamond^1(D).$$

As $u^{(k)} \in H_\diamond^1(D)$ for all $k \in \mathbb{N}$, we have by the Poincaré inequality

$$\|u^{(k)}\| \leq c_1 \|\nabla u^{(k)}\|_2 \leq c_2 \|f\|_2.$$

That is, the sequence $(u^{(k)})_{k \in \mathbb{N}}$ is bounded in $H_\diamond^1(D)$ and we may extract a weakly convergent subsequence, still denoted $(u^{(k)})_{k \in \mathbb{N}}$ for convenience. We have $u^{(k)} \rightharpoonup \tilde{u}$ in $H_\diamond^1(D)$, as $k \rightarrow \infty$, so that it remains to show the convergence of the flows

$$\kappa^{(k)} \nabla u^{(k)} \rightharpoonup \kappa \nabla \tilde{u} \quad \text{in } L^2(D), \text{ as } k \rightarrow \infty.$$

We choose an arbitrary $w \in C_0^\infty(D)$, set $\phi := \nabla \cdot (\kappa \nabla w)$ and consider the solutions $w^{(k)}$, $k \in \mathbb{N}$, of the corresponding homogeneous Dirichlet problem (38). Then we have for every $k \in \mathbb{N}$ the trivial identity

$$\int_D \kappa^{(k)} \nabla w^{(k)} \cdot \nabla u^{(k)} \, dx = \int_D \kappa^{(k)} \nabla u^{(k)} \cdot \nabla w^{(k)} \, dx$$

and the convergence of the flows follows from the compensated compactness lemma [64, Lemma 1.1]. We may thus pass to the limit in the variational formulation to see that $\tilde{u} \equiv u$ is the unique solution to the Neumann problem.

On the other hand, by [54, Lemma 2.2] together with the Hölder continuity up to the boundary of both $p^{(k)}$, $k \in \mathbb{N}$, and p , it follows that for fixed $x \in \overline{D}$, $p^{(k)}(\cdot, x, \cdot) \rightarrow p(\cdot, x, \cdot)$ uniformly on compacts in $(0, T] \times \overline{D}$ for all $T > 0$. It follows from (37) that $u^{(k)}(x) \rightarrow u_\infty(x)$ for all $x \in \overline{D}$ as $k \rightarrow \infty$. Hence, u must coincide with u_∞ and the assertion is proved. \square

Remark 2. Note that the regularization technique we employed in the proof of Theorem 6.4 may be easily modified to prove the Feynman-Kac formula

$$u(x) = \mathbb{E}_x \phi(X_{\tau(D)}), \quad x \in D$$

for the conductivity equation (1) with Dirichlet boundary condition $u|_{\partial D} = \phi$, where $\phi \in H^{1/2}(D)$ and

$$\tau(D) := \inf\{t \geq 0 : X_t \in \mathbb{R}^d \setminus D\}$$

denotes the *first exit time* from the domain D . Such a proof requires the fact that

$$X_{\cdot \wedge \tau^{(k)}(D)}^{(k)} \rightarrow X_{\cdot \wedge \tau(D)} \quad \text{in law on } C([0, \infty); \overline{D})$$

as $n \rightarrow \infty$ for every $x \in D$. This follows immediately from the Lipschitz property of ∂D , implying that all points of ∂D are *regular* in the sense of [35, Chapter 4.2], see also the proof of Theorem 7.1 in the next chapter.

A slight modification of the arguments from above yields the following result which is in fact a corollary rather to the proof of Theorem 6.4 than to its actual statement.

Corollary 1. *Let f be a bounded Borel function and let $\alpha > 0$. Then there is a unique weak solution $u \in C(\overline{D}) \cap H_\diamond^1(D)$ to the boundary value problem*

$$\nabla \cdot (\kappa \nabla u) - \alpha u = 0 \quad \text{in } D$$

subject to the boundary condition (3). This solution admits the Feynman-Kac representation

$$(39) \quad u(x) = \mathbb{E}_x \int_0^\infty e^{-\alpha t} f(X_t) \, dL_t \quad \text{for all } x \in \overline{D}.$$

Proof. Repeat the proof of Theorem 6.4, however, substituting $\{T_t, t \geq 0\}$ with the Feynman-Kac semigroup $\{\tilde{T}_t, t \geq 0\}$, $\tilde{T}_t v(x) := \mathbb{E}_x e^{-\alpha t} v(X_t)$. Note that in contrast to the Neumann problem without the zero-order term, the *gauge function* $\mathbb{E}_x \int_0^\infty e^{-t} \, dL_t$ is finite \mathbb{P}_x -a.s. for every $x \in \overline{D}$. \square

6.2. Complete electrode model. The main result for the complete electrode model (1), (5) is the following theorem.

Theorem 6.5. *For given functions f, g defined by (6) and a voltage pattern $U \in \mathbb{R}^N$ satisfying (7), there is a unique weak solution $u \in C(\overline{D}) \cap H^1(D)$ to the boundary value problem (1), (5). This solution admits the Feynman-Kac representation*

$$(40) \quad u(x) = \mathbb{E}_x \int_0^\infty e_g(t) f(X_t) dL_t \quad \text{for all } x \in \overline{D},$$

with

$$(41) \quad e_g(t) := \exp \left(- \int_0^t g(X_s) dL_s \right), \quad t \geq 0.$$

Before we are ready to give a proof of Theorem 6.5, let us introduce the *Feynman-Kac semigroup* of the complete electrode model, i.e., the one-parameter family of operators $\{T_t^g, t \geq 0\}$ defined by

$$(42) \quad T_t^g v(x) := \mathbb{E}_x e_g(t) v(X_t), \quad x \in \overline{D} \text{ and } t \geq 0.$$

Let us define the *perturbed* Dirichlet form $(\mathcal{E}^g, \mathcal{D}(\mathcal{E}^g))$ by a perturbation of $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ with the measure $g \cdot \sigma$, i.e.,

$$(43) \quad \mathcal{E}^g(v, w) := \mathcal{E}(v, w) + \langle gv, w \rangle_{\partial D}, \quad v, w \in \mathcal{D}(\mathcal{E}^g),$$

where $\mathcal{D}(\mathcal{E}^g) = H^1(D)$ by the standard trace theorem.

Proposition 4. *The Feynman-Kac semigroup $\{T_t^g, t \geq 0\}$ is a strong Feller semigroup on $L^2(D)$.*

Proof. First, it follows from a straightforward computation that $\{T_t^g, t \geq 0\}$ is associated with the Dirichlet form $(\mathcal{E}^g, \mathcal{D}(\mathcal{E}^g))$ and is therefore a strongly continuous sub-Markovian contraction semigroup on $L^2(D)$. T_t^g is a bounded operator from $L^1(D)$ to $L^\infty(D)$ for every $t > 0$, which can be shown using Fatou's lemma. By the Dunford-Pettis theorem, T^g can thus be represented as an integral operator for every $t > 0$,

$$(44) \quad T_t^g \phi(x) = \int_D p^g(t, x, y) \phi(y) dy, \quad \text{for every } \phi \in L^1(D),$$

where for all $t > 0$ we have $p^g(t, x, y) \in L^\infty(D \times D)$ and $p^g(t, x, y) \geq 0$ for a.e. $x, y \in \overline{D}$. In order to prove the strong Feller property, we show that $T_t^g, t > 0$ maps bounded Borel functions to $C(\overline{D})$. As in the papers [32, 49], we use an iterative method to construct the transition kernel density p^g : Let $p_0^g(t, x, y) := p(t, x, y)$ and set

$$p_k^g(t, x, y) := - \int_0^t \int_{\partial D} p(s, x, z) g(z) p_{k-1}^g(t-s, z, y) d\sigma(z) ds, \quad k \in \mathbb{N}.$$

Note that the terms p_k^g are symmetric in the x and y variables by the symmetry of p . By induction using Lemma 6.3 it is not difficult to verify that for all $k \in \mathbb{N}$

$$\int_0^t \int_{\partial D} g(x) p_k^g(s, x, y) d\sigma(x) ds \leq \left(\sup_{x \in \overline{D}} \left\{ \mathbb{E}_x \int_0^t g(X_s) dL_s \right\} \right)^{k+1}$$

and that there is a positive constant c_1 such that

$$(45) \quad |p_k^g(t, x, y)| \leq c_1^{k+1} t^{-d/2} \left(\sup_{x \in \overline{D}} \left\{ \mathbb{E}_x \int_0^t g(X_s) dL_s \right\} \right)^{k+1} \quad \text{for all } k \in \mathbb{N}.$$

Let us show the continuity of p_k^g , $k \in \mathbb{N} \cup \{0\}$. For $k = 0$, this is a consequence of Theorem 1. Now assume that p_{k-1}^g is continuous on $(t_0, T] \times \overline{D} \times \overline{D}$ for $t_0 > 0$, then we have for $t \in (t_0, T]$

$$\begin{aligned} p_k^g(t, x, y) = & - \int_0^{t_0} \int_{\partial D} p(s, x, z) g(z) p_{k-1}^g(t-s, z, y) d\sigma(z) ds \\ & - \int_{t_0}^t \int_{\partial D} p(s, x, z) g(z) p_{k-1}^g(t-s, z, y) d\sigma(z) ds. \end{aligned}$$

Note that the first integral on the right-hand side tends to zero uniformly as $t_0 \rightarrow 0$, which is a consequence of (45), while the second integral is continuous by assumption. Hence, there exists a $T > 0$ such that the series $p^g(t, x, y) := \sum_{k=0}^{\infty} p_k^g(t, x, y)$ converges absolutely and uniformly on any compact subset of $(0, T] \times \overline{D} \times \overline{D}$ and is thus continuous on $(0, T] \times \overline{D} \times \overline{D}$. By the Markov property we have for all $t \in (0, T]$ and every $x \in \overline{D}$ the following expression for $T_t^g \phi(x)$

$$\int_D p^g(t, x, y) \phi(y) dy = \mathbb{E}_x \phi(X_t) + \sum_{k=1}^{\infty} \frac{1}{k!} \mathbb{E}_x \left\{ \left(- \int_0^t g(X_s) dL_s \right)^k \phi(X_t) \right\}.$$

Therefore, the assertion for arbitrary $T > 0$ follows from the Chapman-Kolmogorov equation. \square

Corollary 2. *Let u be defined by the Feynman-Kac formula (40), then $u \in C(\overline{D})$.*

Proof. Let us define a \mathbb{P}_x -martingale by

$$\mathbb{E}_x \left\{ \int_0^{\infty} e_g(s) f(X_s) dL_s | \mathcal{F}_t \right\} = \int_0^t e_g(s) f(X_s) dL_s + e_g(t) u(X_t),$$

where the right-hand side is obtained using the Markov property of X together with the fact that e_g is a multiplicative functional of X . Obviously,

$$e_g(t) u(X_t) - u(x) + \int_0^t e_g(s) f(X_s) dL_s$$

is a \mathbb{P}_x -martingale as well, and hence we have for all $0 \leq s \leq t$

$$e_g(s) u(X_s) = e_g(s) \mathbb{E}_{X_s} e_g(t-s) u(X_{t-s}) + e_g(s) \mathbb{E}_{X_s} \int_0^{t-s} e_g(r) f(X_r) dL_r.$$

Setting $s = 0$ thus yields

$$(46) \quad u(x) = T_t^g u(x) + \mathbb{E}_x \int_0^t e_g(r) f(X_r) dL_r \quad \text{for all } t \geq 0.$$

$T_t^g u$ is continuous on \overline{D} by Proposition 4. To prove that u is continuous on \overline{D} , it is sufficient to show that the second term on the right-hand side of (46) tends to zero uniformly in x , as $t \rightarrow 0$. This is, however, clear since we may estimate

$$\sup_{x \in \overline{D}} \left\{ \mathbb{E}_x \int_0^t e_g(s) f(X_s) dL_s \right\} \leq z^{-1} \max_{l=1, \dots, N} \{U_l\} \sup_{x \in \overline{D}} \{\mathbb{E}_x L_t\},$$

where the right-hand side tends to zero as $t \rightarrow 0$ by Lemma 6.3. \square

The following lemma yields a semimartingale decomposition for the composite process $u(X_t)$ which compensates for the lack of Itô's formula in the proof of Theorem 6.5.

Lemma 6.6. *Let $u \in H^1(D)$ denote the weak solution of the boundary value problem (1), (5). Then for all $t \geq 0$*

$$(47) \quad u(X_t) = u(x) + \int_0^t \nabla u(X_s) dM_s^u - \int_0^t f(X_s) dL_s + \int_0^t g(X_s)u(X_s) dL_s,$$

\mathbb{P}_x -a.s. for q.e. $x \in \overline{D}$.

Proof. Applying the Fukushima decomposition (26) to the perturbed Dirichlet form $(\mathcal{E}^g, H^1(D))$, we obtain the unique decomposition

$$v(X_t^g) - v(X_0^g) = M_t^{g,v} + N_t^{g,v}, \quad \text{for all } t > 0, \quad \mathbb{P}_x\text{-a.s. for q.e. } x \in \overline{D}$$

into a martingale additive functional of finite energy and a continuous additive functional of zero energy of the non-conservative Hunt process X^g associated with $(\mathcal{E}^g, H^1(D))$.

We study the relation between the continuous additive functionals N^v and $N^{g,v}$. Let us assume first that v is in the range of the 1-resolvent associated with the perturbed Dirichlet form, i.e.,

$$v(x) = G_1^g \phi(x) = \mathbb{E}_x \left\{ \int_0^\infty e^{-t - \int_0^t g(X_s) dL_s} \phi(X_t) dt \right\}, \quad x \in \overline{D}$$

for some $\phi \in L^2(D)$. Then we have by the resolvent property, $-\mathcal{L}G_1^g \phi = \phi - v$ so that for all $w \in H^1(D)$

$$\mathcal{E}^g(G_1^g \phi, w) = \int_D (\phi - v)w dx.$$

Using the Revuz correspondence, we see that $N^{g,v}$ admits a semimartingale decomposition, namely

$$N_t^{g,v} = \int_0^t (\phi(X_s^g) - v(X_s^g)) ds.$$

Moreover, an easy computation yields that

$$(48) \quad G_1 \phi - G_1^g \phi = \mathbb{E}_x \left\{ \int_0^\infty e^{-t} g(X_t) G_1^g \phi(X_t) dL_t \right\} \quad \text{for all } \alpha > 0.$$

By Corollary 1, the right-hand side in (48) is the unique weak solution of the elliptic boundary value problem

$$\begin{cases} \nabla \cdot (\kappa \nabla w) - w = 0 & \text{in } D \\ \partial_{\kappa \nu} w = g G_1^g \phi & \text{on } \partial D. \end{cases}$$

That is, it coincides with the 1-potential $U_1(G_1^g \phi(g \cdot \sigma)) \in H^1(D)$. Invoking the Revuz correspondence once more, we see that the zero energy continuous additive functional in the Fukushima decomposition of the 1-potential corresponding to the signed Radon measure $vg \cdot \sigma$ is given by

$$N_t^{U_1(vg \cdot \sigma)} = \int_0^t U_1(vg \cdot \sigma)(X_s) ds - \int_0^t v(X_s)g(X_s) dL_s, \quad \text{for all } t \geq 0.$$

Therefore, we obtain

$$\begin{aligned} N_t^v &= N_t^{G_1 \phi} - N_t^{U_1(vg \cdot \sigma)} \\ &= \int_0^t (\phi(X_s) - G_1 \phi(X_s)) ds - \int_0^t U_1(vg \cdot \sigma)(X_s) ds + \int_0^t v(X_s)g(X_s) dL_s \\ &= \int_0^t (\phi(X_s) - v(X_s)) ds + \int_0^t v(X_s)g(X_s) dL_s. \end{aligned}$$

Moreover, notice that X^g is related to X by a random time change, namely

$$X_s^g = \begin{cases} X_s, & s < \zeta^g \\ \partial, & s \geq \zeta^g, \end{cases}$$

where the lifetime ζ^g is given by

$$\zeta^g := \inf \left\{ t : \int_0^t g(X_s) dL_s > Z \right\}$$

and Z is an exponentially distributed random variable with parameter 1. Hence, we obtain

$$(49) \quad N_t^v = N_t^{g,v} + \int_0^t v(X_s^g) g(X_s^g) dL_s \quad \text{for all } t < \zeta^g.$$

This equality may be generalized to the case of an arbitrary $v \in H^1(D)$ not necessarily in the range of the resolvent using an approximation argument. Namely, we consider the sequence $(v_k)_{k \in \mathbb{N}}$ with $v_k := kG_{k+1}^g v = G_1^k \phi_k$, $\phi_k := k(v - kG_{k+1}^g v)$. Then $v_k \in H^1(D)$ for all $k \in \mathbb{N}$ and the sequence $(v_k)_{k \in \mathbb{N}}$ satisfies both,

$$\lim_{k \rightarrow \infty} \mathcal{E}^g(v_k - v, v_k - v) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \mathcal{E}(v_k - v, v_k - v) = 0$$

so that by [25, Corollary 5.2.1], there exists a subsequence, for convenience still denoted $(v_k)_{k \in \mathbb{N}}$, such that $v_k(X_t^g) \rightarrow v(X_t^g)$, $N_t^{g,v_k} \rightarrow N_t^{g,v}$ and $N_t^{v_k} \rightarrow N_t^v$ uniformly on each finite time interval, \mathbb{P}_x -a.s. for q.e. $x \in \overline{D}$. In particular, it follows that (49) holds for arbitrary $v \in H^1(D)$.

As u solves the boundary value problem (1), (5) we have that

$$\mathcal{E}^g(u, v) = \langle f, v \rangle_{\partial D} \quad \text{for all } v \in H^1(D) \cap C(\overline{D})$$

Since the perturbed Dirichlet form $(\mathcal{E}^g, H^1(D))$ is regular, we obtain from the Revuz correspondence the representation

$$N_t^{g,u} = - \int_0^t f(X_s^g) dL_s, \quad \mathbb{P}_x\text{-a.s. for q.e. } x \in \overline{D}.$$

Finally, using the representation

$$M_t^u = \int_0^t \nabla u(X_s) dM_s^u, \quad \mathbb{P}_x\text{-a.s. for q.e. } x \in \overline{D}$$

for the martingale additive functional, the claim follows from the Fukushima decomposition (26), (49) and the Markov property of X . \square

Proof of Theorem 6.5. There exists a weak solution $u \in H^1(D)$ of the boundary value problem (1), (5) so that with regard to Corollary 2, it remains to show that this weak solution u admits the Feynman-Kac representation (40). Note first that the gauge function

$$(50) \quad \mathbb{E}_x \int_0^\infty e_g(t) dL_t$$

is finite \mathbb{P}_x -a.s. for every $x \in \overline{D}$, hence the expression on the right-hand side of (40) is well-defined. Lemma 6.6 yields the semimartingale decomposition

$$u(X_t) = u(x) + \int_0^t \nabla u(X_s) dM_s^u - \int_0^t f(X_s) dL_s + \int_0^t g(X_s) u(X_s) dL_s,$$

\mathbb{P}_x -a.s. for q.e. $x \in \overline{D}$. Note that the second term on the right-hand side is a local \mathbb{P}_x -martingale and that e_g is continuous, adapted to $\{\mathcal{F}_t, t \geq 0\}$ and of bounded variation. Multiplication by such functions leaves the class of local martingales invariant. Using integration by parts we thus obtain for q.e. $x \in \overline{D}$ the identity

$$u(X_t)e_g(t) = u(x) + \int_0^t e_g(s)\nabla u(X_s) dM_s^u - \int_0^t e_g(s)f(X_s) dL_s,$$

\mathbb{P}_x -a.s., where the second summand on the right-hand side is a local \mathbb{P}_x -martingale. That is, there exists an increasing sequence $(\tau_k)_{k \in \mathbb{N}}$ of stopping times which tend to infinity such that for every $k \in \mathbb{N}$

$$\mathcal{M}_{t \wedge \tau_k} := \int_0^{t \wedge \tau_k} e_g(s)\nabla u(X_s) dM_s^u$$

is a \mathbb{P}_x -martingale. By definition of the term e_g , it is, however, clear that

$$\mathbb{E}_x \sup_{k \in \mathbb{N}} |\mathcal{M}_{t \wedge \tau_k}| < \infty \quad \text{for all } t \geq 0 \text{ and every } x \in \overline{D}$$

which is sufficient for $\{\mathcal{M}_t, t \geq 0\}$ to be a \mathbb{P}_x -martingale by the dominated convergence theorem. Hence,

$$u(x) = \mathbb{E}_x \int_0^t e_g(s)f(X_s) dL_s + \mathbb{E}_x u(X_t)e_g(t) \quad \text{for q.e. } x \in \overline{D}.$$

Letting $t \rightarrow \infty$ finally yields

$$u(x) = \mathbb{E}_x \int_0^\infty e_g(t)f(X_t) dL_t \quad \text{for q.e. } x \in \overline{D},$$

where we have used the fact that u is essentially bounded by standard elliptic regularity theory. Finally, by the fact that we have actually shown in Corollary 2 that the right-hand side in the last equality is continuous up to the boundary, the assertion holds for every $x \in \overline{D}$. \square

Remark 3. Note that the technique we used to prove Theorem 6.5 fails for the Neumann problem corresponding to the continuum model. This comes from the fact that in this case the gauge function (50) becomes infinite. For the same reason Theorem 1.2 from [14], specialized to a zero lower-order term, does not yield the desired Feynman-Kac formula for the continuum model either.

6.3. Mixed boundary value problems. Now we can directly deduce the desired Feynman-Kac formula for the mixed boundary value problem corresponding to the stochastic anomaly detection problem introduced in Section 3.2 of the previous chapter. Recall that in this setting ∂D consists of two disjoint parts $\partial_1 D$ and $\partial_2 D$ and that measurements can be taken only on the *accessible boundary* $\partial_1 D$ while the electric potential vanishes on the *inaccessible boundary* $\partial_2 D$. The deterministic EIT forward problem for the complete electrode model is then given by the conductivity equation (1) subject to the mixed boundary conditions

$$(51) \quad \begin{aligned} \kappa \nu \cdot \nabla u|_{\partial D} + gu|_{\partial D} &= f && \text{on } \partial_1 D \\ u|_{\partial D} &= 0 && \text{on } \partial_2 D. \end{aligned}$$

The following result is a corollary to the line of arguments that led to the proof of Theorem 6.5 rather than to its actual statement.

Corollary 3. *For given functions f, g defined by (6) and a voltage pattern $U \in \mathbb{R}^N$ satisfying (7), there is a unique weak solution $u \in C(\overline{D}) \cap H^1(D)$ to the boundary value problem (1), (51). This solution admits the Feynman-Kac representation*

$$(52) \quad u(x) = \mathbb{E}_x \int_0^\tau e_g(t) f(X_t) dL_t \quad \text{for all } x \in \overline{D},$$

where $\tau := \inf\{t \geq 0 : X_t \in \partial_2 D\}$ denotes the first hitting time of $\partial_2 D$.

Proof. Repeat the computations from Subsection 6.2 with the Feynman-Kac semigroup $\{\tilde{T}_t^g, t \geq 0\}$, where $\tilde{T}_t^g v(x) := \mathbb{E}_x\{[t \leq \tau] e_g(t) v(X_t)\}$ instead of $\{T_t^g, t \geq 0\}$. \square

7. STOCHASTIC HOMOGENIZATION

In this section, we show that the stochastic EIT forward problem may be homogenized both theoretically and numerically by homogenization of the underlying diffusion process on the whole space \mathbb{R}^d . Moreover, we provide a continuum version of a quantitative estimate which has been obtained recently for the discrete random walk in random environment in [27, 19].

7.1. Preliminaries. For convenience of the reader, let us recall some standard concepts from homogenization theory. Let $\phi := (\phi_1, \dots, \phi_d)$, $\phi_i \in L_{\text{loc}}^2(\mathbb{R}^d)$, $i = 1, \dots, d$, denote a vector field. We say that ϕ is a *gradient field* if for every $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \phi_i \partial_j \psi - \phi_j \partial_i \psi dx = 0, \quad i, j = 1, \dots, d.$$

Moreover, we say that ϕ is *divergence-free* if for every $\psi \in C_c^\infty(\mathbb{R}^d)$,

$$\sum_{i=1}^d \int_{\mathbb{R}^d} \phi_i \partial_i \psi dx = 0.$$

Now let us consider a conductivity random field $\{\kappa(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ and let $\{\Theta_x, x \in \mathbb{R}^d\}$ denote the underlying dynamical system which is assumed to satisfy the assumptions (i)-(iv) from Subsection 3.2. A vector field $\phi \in L^2(\Gamma; \mathbb{R}^d)$ is called a *gradient field*, respectively *divergence-free*, if its realizations $\phi(\cdot, \omega) : \mathbb{R}^d \mapsto \mathbb{R}^d$, $x \mapsto \phi(\Theta_x \omega)$ are gradient fields, respectively divergence-free, for \mathcal{P} -a.e. $\omega \in \Gamma$. We define the function spaces

$$\begin{aligned} L_{\text{pot}}^2(\Gamma) &:= \{\phi \in L^2(\Gamma; \mathbb{R}^d) : \phi(\cdot, \omega) \text{ is a gradient field } \mathcal{P}\text{-a.s.}\} \\ L_{\text{sol}}^2(\Gamma) &:= \{\phi \in L^2(\Gamma; \mathbb{R}^d) : \phi(\cdot, \omega) \text{ is divergence-free } \mathcal{P}\text{-a.s.}\}. \end{aligned}$$

If $\phi \in L_{\text{pot}}^2(\Gamma)$, we can find a function $\eta : \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}$ such that $\eta(\cdot, \omega) \in H_{\text{loc}}^1(\mathbb{R}^d)$ and

$$(53) \quad \nabla \eta(\cdot, \omega) = \phi(\Theta \cdot \omega) \quad \text{a.e. in } \mathbb{R}^d \text{ for } \mathcal{P}\text{-a.e. } \omega \in \Gamma.$$

In particular, (53) defines a stationary random field with respect to the measure \mathcal{P} . We call η the *potential* corresponding to ϕ .

Remark 4. Note that $\phi \in L_{\text{pot}}^2(\Gamma)$ does not imply that $\{\phi(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ is a stationary random field with respect to \mathcal{P} . In fact, it can be shown that this is not true for $d = 1$.

We define another function space

$$\mathcal{V}_{\text{pot}}^2 := \{\phi \in L_{\text{pot}}^2(\Gamma) : \mathbb{M}\phi = 0\},$$

so that one obtains an orthogonal Weyl decomposition of $L^2(\Gamma; \mathbb{R}^d)$, namely

$$L^2(\Gamma; \mathbb{R}^d) = \mathcal{V}_{\text{pot}}^2(\Gamma) \oplus L_{\text{sol}}^2(\Gamma),$$

cf., e.g., [64]. Let $\xi \in \mathbb{R}^d$ denote a direction vector, i.e., $|\xi| = 1$. The so-called *auxiliary problem* for the direction ξ reads as follows: Find $\chi^\xi \in \mathcal{V}_{\text{pot}}^2(\Gamma)$ such that $\kappa(\xi + \chi^\xi) \in L_{\text{sol}}^2(\Gamma)$ or equivalently,

$$(54) \quad \mathbb{M}\{\kappa(\xi + \chi^\xi) \cdot \phi\} = 0 \quad \text{for all } \phi \in \mathcal{V}_{\text{pot}}^2(\Gamma).$$

For a proof of existence and uniqueness of the solution to the auxiliary problem we refer the reader to the seminal paper [48] by Papanicolaou and Varadhan.

We can now bring the underlying diffusion processes evolving in the random medium into play by recalling a stochastic homogenization result which was obtained by Lejay [42]. Let $\{\kappa_\varepsilon(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ denote the scaled conductivity random field, see Subsection 3.2, and let $X^{\omega, \varepsilon}$ denote the diffusion process on \mathbb{R}^d which is associated with the regular symmetric Dirichlet form

$$\mathcal{E}^{\omega, \varepsilon}(v, w) := \int_{\mathbb{R}^d} \kappa_\varepsilon(\cdot, \omega) \nabla v \cdot \nabla w \, dx, \quad v, w \in \mathcal{D}(\mathcal{E}^{\omega, \varepsilon}) := H^1(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d)$. It has been shown in [42] that, under assumption (A1) from Subsection 3.2, for \mathcal{P} -a.e. $\omega \in \Gamma$

$$(55) \quad X^{\omega, \varepsilon} \rightarrow X^* \quad \text{in law on } C([0, \infty); \mathbb{R}^d) \text{ as } \varepsilon \rightarrow 0,$$

where X^* denotes the diffusion process on \mathbb{R}^d which is associated with the *homogenized Dirichlet form*

$$\mathcal{E}^*(v, w) := \int_{\mathbb{R}^d} \kappa^* \nabla v \cdot \nabla w \, dx, \quad v, w \in \mathcal{D}(\mathcal{E}^*) := H^1(\mathbb{R}^d)$$

on $L^2(\mathbb{R}^d)$ and the constant, symmetric and positive definite matrix κ^* satisfies the equation

$$(56) \quad \xi \cdot \kappa^* \xi = \mathbb{M}\{(\xi + \chi^\xi) \cdot \kappa(\xi + \chi^\xi)\},$$

where $\chi^\xi \in \mathcal{V}_{\text{pot}}^2(\Gamma)$ denotes the solution to the auxiliary problem (54) for the direction $\xi \in \mathbb{R}^d$.

7.2. Homogenization of the EIT forward problem. The following theorem is our main result. Its assertion is in fact a rather direct consequence of an invariance principle for reflecting diffusion processes obtained recently by Chen, Croydon and Kumagai [13] and the Feynman-Kac formula (52) from Corollary 3.

Theorem 7.1. *Let $\{\kappa_\varepsilon(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ be a stationary random field satisfying assumption (A1) from Subsection 3.2 and assume that the trajectories satisfy $\kappa(\cdot, \omega) \in C_{loc}^{0,1}(\overline{D}; \mathbb{R}^{d \times d})$ or $\kappa(\cdot, \omega)$ piecewise constant for \mathcal{P} -a.e. $\omega \in \Gamma$; let $\Sigma = \emptyset$. Then, for a given voltage pattern $U \in \mathbb{R}^N$, we have for the potentials in the stochastic boundary value problem (11), (12)*

$$(57) \quad u_\varepsilon(x, \omega) \rightarrow u^*(x), \quad x \in \overline{D} \text{ for } \mathcal{P}\text{-a.e. } \omega \in \Gamma, \text{ as } \varepsilon \rightarrow 0,$$

and the corresponding electrode currents satisfy

$$(58) \quad \lim_{\varepsilon \rightarrow 0} J_l(\varepsilon, \omega) = \frac{1}{|E_l|} \int_{E_l} \kappa^* \nu \cdot \nabla u^*(x)|_{\partial_1 D} d\sigma(x) \quad \text{for } \mathcal{P}\text{-a.e. } \omega \in \Gamma,$$

$l = 1, \dots, N$, where the function $u^* \in H_0^1(D \cup \partial_1 D) \cap C(\overline{D})$ is the unique solution to the deterministic forward problem

$$(59) \quad \nabla \cdot (\kappa^* \nabla u^*) = 0 \quad \text{in } D$$

subject to the boundary conditions

$$(60) \quad \begin{aligned} \kappa^* \nu \cdot \nabla u^*|_{\partial_1 D} + g u^*|_{\partial_1 D} &= f && \text{on } \partial_1 D \\ u^*|_{\partial_2 D} &= 0 && \text{on } \partial_2 D \end{aligned}$$

with a constant, symmetric and positive definite matrix κ^* given by (56).

Proof. Let us first show that for \mathcal{P} -a.e. $\omega \in \Gamma$, $u_\varepsilon(x, \omega) \rightarrow u^*(x)$, $x \in \overline{D}$, as $\varepsilon \rightarrow 0$. Let $(\varepsilon_k)_{k \in \mathbb{N}}$ be an arbitrary monotone decreasing null sequence and let $X^{\omega, \varepsilon}$ denote the reflecting diffusion process on the half-space corresponding to the regular symmetric Dirichlet form $(\mathcal{E}^{\omega, \varepsilon}, H^1(\mathbb{R}_-^d \cup \mathbb{R}^{d-1}))$ on $L^2(\mathbb{R}_-^d \cup \mathbb{R}^{d-1})$. By assumption (A1) from Subsection 3.2, we deduce from [13, Section 4] that for \mathcal{P} -a.e. $\omega \in \Gamma$

$$X^{\omega, \varepsilon_k} \rightarrow X^* \quad \text{in law on } C([0, \infty); \mathbb{R}_-^d \cup \mathbb{R}^{d-1}), \text{ as } k \rightarrow \infty,$$

where X^* is the reflecting diffusion process on the half-space associated with the homogenized regular symmetric Dirichlet form $(\mathcal{E}^*, H^1(\mathbb{R}_-^d \cup \mathbb{R}^{d-1}))$ on $L^2(\mathbb{R}_-^d \cup \mathbb{R}^{d-1})$. The constant, symmetric and positive definite matrix κ^* is given by (56).

Let us first show that for every $x \in \overline{D}$ and \mathcal{P} -a.e. realization $\omega \in \Gamma$ of the random medium

$$X_{\cdot \wedge \tau^{\omega, \varepsilon_k}}^{\omega, \varepsilon_k} \rightarrow X_{\cdot \wedge \tau^*}^* \quad \text{in law on } C([0, \infty); \mathbb{R}_-^d \cup \mathbb{R}^{d-1}), \text{ as } k \rightarrow \infty.$$

Consider the functional $F : C([0, \infty); \mathbb{R}_-^d \cup \mathbb{R}^{d-1}) \rightarrow [0, \infty]$,

$$\phi \mapsto \begin{cases} \infty, & \text{if for all } t \geq 0 : |\phi(t)| < R, \\ \inf\{t \geq 0 : |\phi(t)| = R\} & \text{else,} \end{cases}$$

defined in such a way that $F(X^{\omega, \varepsilon_k}) = \tau^{\omega, \varepsilon_k}$. Let $(\phi_k)_{k \in \mathbb{N}}$ denote a sequence of continuous functions that converges uniformly towards ϕ on compacts in $[0, \infty)$. If $\liminf_{k \rightarrow \infty} F(\phi_k)$ is finite, then we may extract a subsequence, still denoted $(\phi_k)_{k \in \mathbb{N}}$ for convenience, such that $F(\phi_k) \rightarrow \liminf_{k \rightarrow \infty} F(\phi_k)$. We have

$$\begin{aligned} |\phi(\liminf_{k \rightarrow \infty} F(\phi_k)) - \phi_k(F(\phi_k))| &\leq |\phi(\liminf_{k \rightarrow \infty} F(\phi_k)) - \phi(F(\phi_k))| \\ &\quad + |\phi(F(\phi_k)) - \phi_k(F(\phi_k))| \end{aligned}$$

and by our assumption, the right-hand side vanishes as $k \rightarrow \infty$. From the closedness of $\partial_1 D$, we conclude hence that $\phi(\liminf_{k \rightarrow \infty} F(\phi_k)) \in \partial_1 D$. In particular we have that $F(\phi) \leq \liminf_{k \rightarrow \infty} F(\phi_k)$ if $\sup_{t \in [0, T]} |\phi(t) - \phi_k(t)| \rightarrow 0$, as $t \rightarrow \infty$, for all $T > 0$, i.e., F is lower semi-continuous. Now assume that ϕ is a discontinuity point of F , i.e.,

there is $\delta > 0$ and k_0 such that $F(\phi) + \delta \leq F(\phi_k) < \infty$ for all $k \geq k_0$. Then it follows that $\phi(t) \in \overline{D}$ for all $t \in [F(\phi), \liminf_{k \rightarrow \infty} F(\phi_k))$. However, the boundary $\partial_1 D$ is regular in the sense of [35, Chapter 4.2], that is, a diffusion process originating from $x \in \partial_1 D$ will immediately exit from \overline{D} $\mathbb{P}_x^{\omega, \varepsilon_k}$ -a.s., respectively \mathbb{P}_x^* -a.s. In other words, the set of discontinuities of F is a null set with respect to these measures and hence the claim follows from the continuous mapping theorem, cf. [8]. Now, as in the proof of [53, Theorem 5.1], it follows with the Fukushima decompositions in Section 7 of the second chapter that

$$(X_{\cdot \wedge \tau^{\omega, \varepsilon_k}}^{\omega, \varepsilon_k}, L_{\cdot \wedge \tau^{\omega, \varepsilon_k}}^{\omega, \varepsilon_k}) \rightarrow (X_{\cdot \wedge \tau^*}^*, L_{\cdot \wedge \tau^*}^*) \quad \text{in law on } C([0, \infty); \mathbb{R}_-^d \cup \mathbb{R}^{d-1} \times \mathbb{R}_+),$$

as $k \rightarrow \infty$. With regard to the Feynman-Kac formula (52), the assertion (57) can be proved as follows:

Suppose first that f is continuous. Then we have to show that

$$\limsup_{j \rightarrow \infty} \left| \mathbb{E}_x \int_0^{\tau_j} e_g^{\omega, \varepsilon_j}(t) f(X_t^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} - \mathbb{E}_x \int_0^{\tau} e_g(t) f(X_t) dL_t \right| = 0.$$

We estimate this with a difference of truncated Riemann sums

$$\left| \sum_{k=0}^{\lfloor N/h \rfloor} \left(\mathbb{E}_x \int_{t_k}^{t_{k+1}} e_g^{\omega, \varepsilon_j}(t_k) f(X_{t_k}^{\omega, \varepsilon_j}) [t_{k+1} < \tau_j] dL_t^{\omega, \varepsilon_j} - \mathbb{E}_x \int_{t_k}^{t_{k+1}} e_g(t_k) f(X_{t_k}) [t_{k+1} < \tau] dL_t \right) \right|,$$

where $t_k = kh$ and h is a step size. The difference $S_{N,h}$ of truncated Riemann sums for fixed N and h goes to zero as $j \rightarrow \infty$ if we assume that $(X^{\omega, \varepsilon_j}, L^{\omega, \varepsilon_j}, A^{\omega, \varepsilon_j})$ converge weakly to (X, L, A) , where $A^{\omega, \varepsilon_j} = \log e_g^{\omega, \varepsilon_j}$ and $A = \log e_g$.

The error terms for $X^{\omega, \varepsilon_j}$ and X are nearly analogous, so it is enough to consider just $X^{\omega, \varepsilon_j}$ in detail. The increments (excluding the edge case where $t_k < \tau_j < t_{k+1}$ which we will omit but which can be treated in the same way) are of following form

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} (e_g^{\omega, \varepsilon_j}(t) f(X_t^{\omega, \varepsilon_j}) - e_g^{\omega, \varepsilon_j}(t_k) f(X_{t_k}^{\omega, \varepsilon_j})) dL_t^{\omega, \varepsilon_j} \\ &= \int_{t_k}^{t_{k+1}} (e_g^{\omega, \varepsilon_j}(t) - e_g^{\omega, \varepsilon_j}(t_k)) f(X_t^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} \\ & \quad + e_g^{\omega, \varepsilon_j}(t_k) \int_{t_k}^{t_{k+1}} (f(X_t^{\omega, \varepsilon_j}) - f(X_{t_k}^{\omega, \varepsilon_j})) dL_t^{\omega, \varepsilon_j} \end{aligned}$$

The latter term can be handled with the continuity of the paths of X together with the uniform continuity of f on the compact set \overline{D} . This is since

$$\begin{aligned} & \left| e_g^{\omega, \varepsilon_j}(t_k) \int_{t_k}^{t_{k+1}} (f(X_t^{\omega, \varepsilon_j}) - f(X_{t_k}^{\omega, \varepsilon_j})) dL_t^{\omega, \varepsilon_j} \right| \\ & \leq \left(2(1 - \psi_\delta(\theta_h(X^{\omega, \varepsilon_j}))) \|f\|_\infty + \theta_{2\delta}(f) \psi_\delta(\theta_h(X^{\omega, \varepsilon_j})) \right) (L_{t_{k+1}}^{\omega, \varepsilon_j} - L_{t_k}^{\omega, \varepsilon_j}), \end{aligned}$$

where $\theta_\delta(x)$ is the maximum variation of the function x on the interval $[0, N]$

$$\theta_\delta(x) := \sup \{ |x(t) - x(s)| ; 0 \leq t, s \leq N, |t - s| < \delta \}$$

and $\psi_\delta(t)$ is the continuous approximation of the indicator function $[t < \delta]$ with support in $[-2\delta, 2\delta]$. Therefore, after taking the limit $j \rightarrow \infty$, the latter terms give that the

corresponding total approximation error can be bounded by

$$4\|f\|_\infty \mathbb{E}_x\{[\theta_h(X) \geq \delta]L_{\tau \wedge N}\} + 2\theta_{2\delta}(f)\mathbb{E}_x\{L_{\tau \wedge N}\}$$

which goes to zero as $\delta \rightarrow 0$.

The first term can be estimated by

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} (e_g^{\omega, \varepsilon_j}(t) - e_g^{\omega, \varepsilon_j}(t_k)) f(X_t^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} \\ & \leq 2\|f\|_\infty \|g\|_\infty \int_{t_k}^{t_{k+1}} (L_t^{\omega, \varepsilon_j} - L_{t_k}^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} = \|f\|_\infty \|g\|_\infty (L_{t_{k+1}}^{\omega, \varepsilon_j} - L_{t_k}^{\omega, \varepsilon_j})^2 \end{aligned}$$

since $g \geq 0$. Therefore, after taking $j \rightarrow \infty$ the first term gives the total approximation error that is bounded by

$$2\|f\|_\infty \|g\|_\infty \left(\mathbb{E}_x\{[\theta_h(L) \geq \delta]L_{\tau \wedge N}\} + 2\delta \mathbb{E}_x\{L_{\tau \wedge N}\} \right)$$

which goes to zero as $\delta \rightarrow 0$ as well.

We are still left with the truncation. The truncation can be removed since τ_j and τ are a.s. finite and moreover, $L_{\tau_j}^{\omega, \varepsilon_j}$ converges weakly to L_τ and thus, we get a uniform estimate

$$\begin{aligned} & \limsup_{j \rightarrow \infty} \left| \mathbb{E}_x \int_0^{\tau_j} e_g^{\omega, \varepsilon_j}(t) f(X_t^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} - \int_0^\tau e_g(t) f(X_t) dL_t \right| \\ & \leq 2\|f\|_\infty \mathbb{E}_x\{L_\tau - L_{\tau \wedge N}\} \end{aligned}$$

which gives the claimed convergence for continuous f if we can prove weak convergence of the sequence $(A^{\omega, \varepsilon_j})_{j \in \mathbb{N}}$.

Therefore, we will next verify the weak convergence of $A^{\omega, \varepsilon_j}$ jointly with $(X^{\omega, \varepsilon_j}, L^{\omega, \varepsilon_j})$. We can assume that $X^{\omega, \varepsilon_j} \rightarrow X$ and $L^{\omega, \varepsilon_j} \rightarrow L$ almost surely in $C(0, T)$. Suppose that $g = \sum_{l=1}^N g_l [E_l]$ where $g_l \geq 0$ and continuous, $E_l \cap E_k = \emptyset$ and $\sigma(\partial E_l) = 0$. Therefore, it is enough to show the convergence for $g = [E_l]$ for E_l open in ∂D . Since E_l is open, it can be approximated from below by an increasing sequence of continuous functions g'_k that converge pointwise to g .

It follows from the almost sure convergence of $X^{\omega, \varepsilon_j}$ and $L^{\omega, \varepsilon_j}$ that

$$\liminf_{j \rightarrow \infty} \int_0^t g(X_t^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} \geq \limsup_{k \rightarrow \infty} \int_0^t g'_k(X_t) dL_t = \int_0^t g(X_t) dL_t.$$

Since $X^{\omega, \varepsilon_j}$ and X will not hit ∂E_l in $[0, T]$ almost surely, we can get the other direction by considering $1 - g$. Therefore,

$$\limsup_{j \rightarrow \infty} \int_0^t g(X_t^{\omega, \varepsilon_j}) dL_t^{\omega, \varepsilon_j} \leq \int_0^t g(X_t) dL_t.$$

Using countability of the rational numbers, we obtain

$$\forall t \in [0, T] \cap \mathbb{Q}: \lim_{j \rightarrow \infty} A^{\omega, \varepsilon_j}(t) = A(t)$$

almost surely and by monotonicity this implies the almost sure convergence of $A^{\omega, \varepsilon_j}$ to A in $C(0, T)$, which implies the weak convergence of the original versions. The same technique allows us to extend the assertion to the case of discontinuous f .

For the proof of (58) note that the boundary condition (12) allows us to write

$$(61) \quad J_l(\varepsilon_k, \omega) = \frac{1}{|E_l|} \int_{E_l} (f - gu_{\varepsilon_k}(\cdot, \omega)|_{\partial_1 D}) d\sigma(x) \quad \text{for } \mathcal{P}\text{-a.e. } \omega \in \Gamma$$

and that (12) may be written in the form $(\Lambda_{\kappa_\varepsilon} + gI)u_\varepsilon = f$. We deduce from the well-posedness of the forward problem that $(\Lambda_{\kappa_\varepsilon} + gI)^{-1}, L^2(\partial D) \rightarrow L^2(\partial D)$, is bounded. Since, due to assumption (A1), the corresponding constant does not depend on ε , the sequence $(u_{\varepsilon_k})_{k \in \mathbb{N}}$ is bounded in $L^2(\partial D)$ for \mathcal{P} -a.e. $\omega \in \Gamma$ implying its uniform integrability, see, e.g., [41]. As we already know the pointwise convergence $u_{\varepsilon_k}(x, \omega) \rightarrow u^*(x)$, $x \in \overline{D}$, as $k \rightarrow \infty$, an application of Egorov's theorem yields convergence in $L^1(\partial D)$ so that the assertion follows by the triangle inequality and taking the limit $k \rightarrow \infty$ inside the integration in (61). \square

Remark 5. We would like to point out that the effective conductivity κ^* is determined by the invariance principle on the whole space \mathbb{R}^d .

Remark 6. We would like to mention that for more general functions f we can prove that the well-posedness of the forward problem (11), (12) together with the weak convergence of $(X^{\omega, \varepsilon}, L^{\omega, \varepsilon}, A^{\omega, j})$ and some stopping arguments give a slightly weaker form of (57), namely that the convergence holds in \mathcal{P} -probability.

7.3. Continuum approximation of the effective conductivity. In this subsection, we provide the theoretical foundation for the convergence analysis of numerical homogenization methods based on simulation of the underlying diffusion process. More precisely, a rigorous convergence analysis of such a method requires a quantitative estimate that is stronger than the qualitative result (55), which was obtained in [42] using merely the central limit theorem for martingales. We provide such a quantitative result in the following theorem by generalizing a classical argument due to Kipnis and Varadhan [40]. The proof relies on new spectral bounds which were obtained recently by Gloria and Otto [29]. We refer the reader to the recent papers [19, 27] for an analogous estimate for the discrete lattice random walk in random environment as well as to the paper [44] which was the first one to use the Kipnis and Varadhan argument in order to obtain quantitative results.

Theorem 7.2. *Let $\{\kappa_\varepsilon(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ be a stationary random field satisfying assumptions (A1) and (A2) from Subsection 3.2. Then for every direction vector $\xi \in \mathbb{R}^d$ there exist positive constants c_1, c_2 such that*

$$(62) \quad \left| \frac{\overline{\mathbb{E}}(X_t \cdot \xi)^2}{2t} - \xi \cdot \kappa^* \xi \right| \leq c_1 \begin{cases} |\log t|^{c_2} t^{-1} & d = 2 \\ t^{-1} & d = 3. \end{cases}$$

Proof. For fixed $\omega \in \Gamma$ let us consider the diffusion process X^ω on \mathbb{R}^d which is associated with the symmetric regular Dirichlet form $(\mathcal{E}^{\omega, 1}, H^1(\mathbb{R}^d))$ on $L^2(\mathbb{R}^d)$ under the measure \mathbb{P}_0^ω . Following [40], we search for a decomposition of the form

$$(63) \quad X_t^\omega = M_t^\omega + R_t^\omega,$$

where M_t^ω is a \mathbb{P}_0^ω -martingale and for every direction $\xi \in \mathbb{R}^d$ the projected remainder $R_t^\omega \cdot \xi$ converges to zero in $L^2(\overline{\Gamma})$ as $t \rightarrow \infty$.

Once we have found a suitable decomposition (63), we first show that

$$(64) \quad t^{-1} \overline{\mathbb{E}}\{(X_t \cdot \xi)^2 - (M_t \cdot \xi)^2\} = t^{-1} \overline{\mathbb{E}}(R_t \cdot \xi)^2.$$

Then we will use spectral calculus to estimate the right-hand side of (64).

In order to obtain the decomposition (63), we recall that the auxiliary problem (54) is equivalent to the following stochastic elliptic equation in the physical space \mathbb{R}^d : Find

$\chi^\xi \in \mathcal{V}_{\text{pot}}^2(\Gamma)$ such that for \mathcal{P} -a.e. $\omega \in \Gamma$, the corresponding potential $\eta^\xi : \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}$ is in $C(\mathbb{R}^d) \cap H_{\text{loc}}^1(\mathbb{R}^d)$ as a function of x , satisfies $\eta^\xi(0, \omega) = 0$ and

$$(65) \quad -\nabla \cdot \kappa(\cdot, \omega)(\xi + \nabla \eta^\xi(\cdot, \omega)) = 0 \quad \text{in } \mathbb{R}^d.$$

Let us define the function

$$\phi : \mathbb{R}^d \times \Gamma \rightarrow \mathbb{R}^d, \quad \phi(x, \omega) := x + \eta(x, \omega) - \eta(0, \omega),$$

where $\eta := (\eta^{e_1}, \dots, \eta^{e_d})^T$ and η^{e_i} , $i = 1, \dots, d$, denotes the potential corresponding to the solution to the auxiliary problems for the coordinate directions. As the transition density kernel of X^ω is jointly Hölder-continuous and $\phi_i(\cdot, \omega) \in C(\mathbb{R}^d) \cap H_{\text{loc}}^1(\mathbb{R}^d)$, $i = 1, \dots, d$, for \mathcal{P} -a.e. $\omega \in \Gamma$, the Fukushima decomposition of $\phi_i(X_t^\omega, \omega)$ holds for every starting point $x \in \mathbb{R}^d$ rather than quasi-every $x \in \mathbb{R}^d$. A straightforward computation using the fact that η is defined via the auxiliary problem yields that the continuous additive functional of zero energy in the Fukushima decomposition vanishes so that

$$\phi_i(X_t^\omega, \omega) = M_t^{\phi_i(\cdot, \omega)}, \quad i = 1, \dots, d.$$

We set

$$M_t^\omega := (M_t^{\phi_1(\cdot, \omega)}, \dots, M_t^{\phi_d(\cdot, \omega)})^T \quad \text{and} \quad R_t^\omega := -\eta(X_t^\omega, \omega) + \eta(0, \omega)$$

and consider the quantity

$$(66) \quad \mathbb{E}(X_t \cdot \xi)^2 = \mathbb{E}(M_t \cdot \xi)^2 + \mathbb{E}(R_t \cdot \xi)^2 + 2\mathbb{E}(M_t \cdot \xi)(R_t \cdot \xi).$$

By computing the predictable quadratic variation of the martingale additive functional we obtain for all $t \geq 0$ and a.e. $x \in \mathbb{R}^d$

$$\mathbb{E}_x^\omega(M_t^\omega \cdot \xi)^2 = \mathbb{E}_x^\omega \left(\int_0^t 2(\xi + \nabla \eta^\xi(X_s, \omega)) \cdot \kappa(X_s, \omega)(\xi + \nabla \eta^\xi(X_s, \omega)) ds \right).$$

By the stationarity of $\{\nabla \eta^\xi(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ with respect to \mathcal{P} we have thus

$$\mathbb{E}(M_t^\omega \cdot \xi)^2 = 2\xi \cdot \kappa^* \xi t \quad \text{for all } t \geq 0.$$

Moreover, as in [18], it follows that the last term on the right-hand side of (66) vanishes. This can be seen by studying the so-called *environment seen by the particle* process, i.e., the stochastic process defined by

$$Y_t^\omega := \begin{cases} \Theta_{X_t^\omega} \omega & t > 0 \\ \omega & t = 0. \end{cases}$$

$\{Y_t^\omega, t \geq 0\}$ is a stationary process with respect to the annealed measure $\overline{\mathbb{P}}$, i.e., for every finite collection of times $t^{(i)}$, $i = 1, \dots, k$, the joint distribution of $Y_{t^{(1)}+h}, \dots, Y_{t^{(k)}+h}$ under $\overline{\mathbb{P}}$ does not depend on $h \geq 0$. It is well known that the underlying dynamical system $\{\Theta_x, x \in \mathbb{R}^d\}$ defines a d -parameter group $\{\mathbf{S}_x, x \in \mathbb{R}^d\}$ of unitary operators on $L^2(\Gamma)$ by $\mathbf{S}_x \psi(\omega) := \psi(\Theta_x \omega)$ and this group is strongly continuous, cf. [42]. Its d infinitesimal generators $(\mathbf{D}_1, \mathcal{D}(\mathbf{D}_1)), \dots, (\mathbf{D}_d, \mathcal{D}(\mathbf{D}_d))$ are given by

$$\mathbf{D}_i \psi = \lim_{h \rightarrow 0+} \frac{\mathbf{S}_{he_i} \psi - \psi}{h}, \quad i = 1, \dots, d,$$

for all $\psi \in L^2(\Gamma)$ such that the limit exists. These operators are closed and densely defined. We denote $\mathbf{D} := (\mathbf{D}_1, \dots, \mathbf{D}_d)^T$ and introduce the infinitesimal generator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ on $L^2(\Gamma)$ of the environment viewed by the particle process, that is, the non-negative definite self-adjoint operator $\mathcal{L} := -\mathbf{D} \cdot \kappa \mathbf{D}$ on $L^2(\Gamma)$. Note that due to the fact that the trajectories of the random conductivity field are in general not

differentiable, one has to use Dirichlet form theory in order to give a precise meaning to computations involving \mathcal{L} . We refer the reader to the work [42] where this has been carried out in some detail. By the self-adjointness of \mathcal{L} , the law of the environment as viewed from the particle process under $\bar{\mathbb{P}}$ is invariant with respect to time reversal and M^ω is odd by [21, Corollary 2.1], i.e., it changes its sign under time reversal, whereas R^ω , which is the zero energy part of the Fukushima decomposition (63), is even by [21, Theorem 2.1]. Thus, we notice that the identity (64) holds, as claimed.

We will next show that the estimate (62) follows from the following spectral representation

$$(67) \quad \bar{\mathbb{E}}(R_t \cdot \xi)^2 = 2 \int_0^\infty (1 - e^{-\lambda t}) \lambda^{-2} d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi),$$

where $\{E_\lambda, \lambda \in \mathbb{R}\}$ is the unique spectral family given by the spectral theorem such that $\mathcal{L} = \int_0^\infty \lambda dE_\lambda$ and the function $\mathbf{v}^\xi := \mathbf{D} \cdot \kappa \xi \in L^2(\Gamma)$. Indeed, given the formula (67) for the projected remainder and due to the assumption (A2), we can now exploit the following optimal estimate from [28, 29]: For all $0 < \gamma \leq 1$, there exists a positive constant c such that

$$\int_0^\gamma d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) \leq c \gamma^{d/2+1}.$$

More precisely, we split the integral (67) into three parts, the first ranging from 0 to t^{-1} , the second from t^{-1} to 1 and the third from 1 to ∞ , respectively when $t > 1$. For the latter we have the trivial estimate

$$\int_1^\infty d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) \leq \int_0^\infty d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) = \mathbb{M}\{(\mathbf{v}^\xi)^2\}.$$

The first part is bounded by a positive constant as well, namely by

$$\begin{aligned} \int_0^{t^{-1}} t \lambda^{-1} d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) &= t \int_0^{t^{-1}} \int_\lambda^\infty \alpha^{-2} d\alpha d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) \\ &= t \int_0^\infty \alpha^{-2} \int_0^{\alpha \wedge t^{-1}} d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) \\ &\leq ct \int_0^\infty \alpha^{-2} (\alpha \wedge t^{-1})^{d/2+1} d\alpha. \end{aligned}$$

Similarly, the second part can be estimated by

$$\begin{aligned} \int_{t^{-1}}^1 \lambda^{-2} d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) &= 2 \int_{t^{-1}}^1 \int_\lambda^\infty \alpha^{-3} d\alpha d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi) \\ &\leq 2c \int_{t^{-1}}^\infty \alpha^{-3} (\alpha \wedge 1)^{d/2+1} d\alpha, \end{aligned}$$

which diverges logarithmically for $d = 2$ and is bounded by a positive constant for $d = 3$. Therefore, combining these computations with the identities (64) and (67) the estimate (62) follows.

It remains to prove the spectral representation (67). In order to take advantage of the spectral theorem, we would like to express the projected remainder in the form $\mathbb{M}\psi_1(\mathcal{L})\mathbf{v}^\xi\psi_2(\mathcal{L})\mathbf{v}^\xi$ for some bounded continuous functions ψ_1 and ψ_2 . However, for this we would need the components of η to be stationary, which is not the case. Furthermore, we would want to use $\psi_1(x) = x^{-1}$ which is unbounded at zero.

Inspired by the computations in [48], we can find a remedy for both of these obstructions; namely we consider the function

$$R_t^{\omega, \delta} := -\eta_\delta(X_t^\omega, \omega) + \eta_\delta(0, \omega),$$

where η_δ is defined in analogy to η with the difference that it corresponds to a different auxiliary problem, modified by a zero order term. This modified auxiliary problem reads as follows: Find $\eta_\delta^\xi(\cdot, \omega) \in C(\mathbb{R}^d) \cap H_{\text{loc}}^1(\mathbb{R}^d)$, $\delta > 0$ such that for \mathcal{P} -a.e. $\omega \in \Gamma$, the random field $\{\eta_\delta^\xi(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ is stationary with respect to \mathcal{P} with $\mathbb{M}\eta_\delta^\xi(x, \cdot) = 0$ for every $x \in \mathbb{R}^d$ and satisfies

$$\delta\eta_\delta^\xi(\cdot, \omega) - \nabla \cdot \kappa(\cdot, \omega)(\xi + \nabla\eta_\delta^\xi(\cdot, \omega)) = 0 \quad \text{in } \mathbb{R}^d$$

for \mathcal{P} -a.e. $\omega \in \Gamma$. We refer to [48] for the proof of existence and uniqueness of η_δ^ξ , $\delta > 0$. Note that $\eta_\delta^\xi(0, \omega) \neq 0$ in general so that we have for the projected modified remainder the expression

$$(68) \quad \overline{\mathbb{E}}(R_t^{\cdot, \delta} \cdot \xi)^2 = \overline{\mathbb{E}}(\eta_\delta^\xi(X_t)) - 2\overline{\mathbb{E}}\eta_\delta^\xi(X_t, \cdot)\eta_\delta^\xi(0, \cdot) + \overline{\mathbb{E}}(\eta_\delta^\xi(0, \cdot))^2.$$

The equivalent formulation on $L^2(\Gamma)$ of the modified auxiliary problem for the direction $\xi \in \mathbb{R}^d$ reads as follows: Find the unique solution $\eta_\delta^\xi \in L^2(\Gamma)$ of the elliptic equation

$$\delta\eta_\delta^\xi - \mathbf{D} \cdot \kappa(\xi + \mathbf{D}\eta_\delta^\xi) = 0 \quad \text{in } \Gamma.$$

In particular, the function \mathbf{v}^ξ was chosen such that $\mathbf{v}^\xi = (\delta + \mathcal{L})\eta_\delta^\xi = \mathcal{L}\eta^\xi$.

We will now only need to show that the modified remainder for fixed δ and fixed t can be written in the form $\mathbb{M}\psi_1(\mathcal{L})\mathbf{v}^\xi\psi_2(\mathcal{L})\mathbf{v}^\xi$, where $\psi_1(x) = 2(x + \delta)^{-1}$ and $\psi_2(x) = 2e^{-tx}(x + \delta)^{-1}$. In fact, the introduction of the zero order perturbation removes the singularity coming from the term x^{-1} .

The first term on the right-hand side of (68) may equivalently be written as

$$\overline{\mathbb{E}}(\eta_\delta^\xi(X_t, \cdot))^2 = \overline{\mathbb{E}}(\eta_\delta^\xi(0, Y_t))^2 = \mathbb{M}(\eta_\delta^\xi(0, \cdot))^2,$$

where we have used the stationarity of the environment as viewed from the particle with respect to $\overline{\mathbb{P}}$ and the stationarity of the random field $\{\eta_\delta^\xi(x, \omega), (x, \omega) \in \mathbb{R}^d \times \Gamma\}$ with respect to \mathcal{P} , respectively. Therefore, treating the second term on the right-hand side of (68) we obtain

$$(69) \quad \overline{\mathbb{E}}(R_t^{\cdot, \delta} \cdot \xi)^2 = 2\mathbb{M}(\eta_\delta^\xi(0, \cdot))^2 - 2\mathbb{M}\eta_\delta^\xi(0, \cdot)T_t\eta_\delta^\xi(0, \cdot),$$

where $\{T_t^\omega, t \geq 0\}$ denotes the strongly continuous semigroup on $L^2(\mathbb{R}^d)$ associated with $(\mathcal{E}^{\omega, 1}, H^1(\mathbb{R}^d))$ which satisfies

$$\mathbb{E}_0^\omega\eta_\delta^\xi(X_t^\omega, \omega) = T_t^\omega\eta_\delta^\xi(0, \omega).$$

Going back to (69), respectively the corresponding identity on $L^2(\Gamma)$, we have thus for the first term on the right-hand side

$$2\mathbb{M}(\eta_\delta^\xi)^2 = 2\mathbb{M}(\delta + \mathcal{L})^{-1}\mathbf{v}^\xi(\delta + \mathcal{L})^{-1}\mathbf{v}^\xi = 2 \int_0^\infty (\delta + \lambda)^{-2} d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi),$$

whereas the second term may be written in the form

$$2\mathbb{M}(\delta + \mathcal{L})^{-1}\mathbf{v}^\xi e^{-t\mathcal{L}}(\delta + \mathcal{L})^{-1}\mathbf{v}^\xi = 2 \int_0^\infty (\delta + \lambda)^{-2} e^{-t\lambda} d(E_\lambda \mathbf{v}^\xi, \mathbf{v}^\xi).$$

Altogether we have obtained

$$\mathbb{E}(R_t^{\cdot, \delta} \cdot \xi)^2 = 2 \int_0^\infty (1 - e^{-\lambda t})(\delta + \lambda)^{-2} d(E_\lambda v^\xi, v^\xi)$$

and the right-hand side of this equality converges as $\delta \rightarrow 0$ so that the spectral representation (67) holds. \square

We would like to remark that it appears that the spectral properties of operators with stationary coefficients and the stationarity of the corresponding fields (as in the proof of Theorem 7.2) might be very closely related in general. However, analysing this potential connection that is touched by the work [48] is left for the future studies.

8. CONCLUSION

We have derived Feynman-Kac formulae for the forward problem of electrical impedance tomography and studied the interconnection between these formulae and stochastic homogenization. Using the properties of the underlying diffusion processes and some new spectral estimates from [27, 28] we have then obtained a bound on the speed of convergence of the projected mean-square displacement of the processes. These results provide the theoretical foundation for the development of new scalable continuum Monte Carlo homogenization schemes.

Both, the homogenization of the forward model for the complete electrode model and the stochastic numerical approximation of the effective conductivity have direct applications in EIT anomaly detection problems for random heterogeneous background media, cf. [55].

ACKNOWLEDGMENTS

The research of M. Simon was supported by the Deutsche Forschungsgemeinschaft (DFG) under grant HA 2121/8 -1 583067. This work is part of M. Simon's Ph.D thesis, who would like to express his gratitude to his advisor Prof. Martin Hanke for his guidance and continuous support. He would also like to thank Prof. Lassi Päivärinta for the kind invitation to the Department of Mathematics and Statistics at the University of Helsinki, where part of the work was carried out. The research of P. Piironen was supported by Academy of Finland (AF) under Finnish Centre of Excellence in Inverse Problems Research 2012–2017, decision number 250215. He has also been supported by an AF project, decision number 141075. Both authors would like to thank Prof. Antoine Gloria and Prof. Elton Hsu for carefully reading parts of this manuscript and for their insightful comments and suggestions.

REFERENCES

- [1] W. Arendt and N. Nikolski. *Vector-valued holomorphic functions revisited*, Math. Z., **234** (2000), 777–805.
- [2] M. Avellaneda, Th. Y. Hou and G. C. Papanicolaou. *Finite difference approximations for partial differential equations with rapidly oscillating coefficients*, RAIRO Modél. Math. Anal. Numér., **25** (1991), 693–710.
- [3] I. Babuska, R. Tempone and G. E. Zouraris. *Solving elliptic boundary value problems with uncertain coefficients by the finite element method: the stochastic formulation*, Comput. Methods Appl. Mech. Engrg., **194** (2005), 1251–1294.
- [4] G. Barles, F. Da Lio, P.-L. Lions and P. E. Souganidis. *Ergodic problems and periodic homogenization for fully non-linear equations in half-space type domains with Neumann boundary conditions*, Indiana Univ. Math. J., **57** (2008), 2355–2376.

- [5] R. F. Bass and P. Hsu, *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains*, Ann. Probab., **19** (1991), 486–508.
- [6] A. Bensoussan, J. L. Lions and G. Papanicolaou. *Asymptotic analysis for periodic structures*, AMS Chelsea Publishing, Providence, RI, 1978.
- [7] S. A. Berggren, D. Lukkassen, A. Meidell and L. Simula. *A new method for numerical solution of checkerboard fields*, J. Appl. Math., **1** (2001), 157–173.
- [8] P. Billingsley. *Probability and measure*, John Wiley & Sons Inc., New York, 1995.
- [9] G. A. Brosamler, *A probabilistic solution of the Neumann problem*, Math. Scand., **38** (1976), 137–147.
- [10] A. Benchérif-Madani and É. Pardoux, *A probabilistic formula for a Poisson equation with Neumann boundary condition*, Stoch. Anal. Appl., **27** (2009), 739–746.
- [11] A. Bourgeat and A. Piatnitski. *Approximations of effective coefficients in stochastic homogenization*, Ann. Inst. H. Poincaré Probab. Statist., **40** (2004), 153–165.
- [12] E.A. Carlen, S. Kusuoka and D. W. Stroock, *Upper bounds for symmetric Markov transition functions*, Ann. Inst. H. Poincaré Probab. Statist., **23** (1987), 245–287.
- [13] Z. Q. Chen, D. A. Croydon and T. Kumagai. *Quenched invariance principles for random walks and elliptic diffusions in random media with boundary*, to appear in Ann. Probab.
- [14] Z. Q. Chen and T. Zhang, *Time-reversal and elliptic boundary value problems*, Ann. Probab., **37** (2009), 1008–1043.
- [15] Z. Q. Chen and T. Zhang. *A probabilistic approach to mixed boundary value problems for elliptic operators with singular coefficients*, Proc. Amer. Math. Soc., **142** (2014), 2135–2149.
- [16] M. Cheney, D. Isaacson and J. C. Newell. *Electrical impedance tomography*, SIAM Rev., **41** (1999), 85–10.
- [17] M. Deaconu and A. Lejay. *A random walk on rectangles algorithm*, Methodol. Comput. Appl. Probab., **8** (2006), 135–151.
- [18] A. De Masi, P.A. Ferrari, S. Goldstein and W. D. Wick. *An invariance principle for reversible Markov processes. Applications to random motions in random environments*, J. Stat. Phys., **55** (1988), 787–855.
- [19] A.-C. Egloffé, A. Gloria., J.-C. Mourrat and T. N. Nguyen. *Random walk in random environment, corrector equation, and homogenized coefficients: from theory to numerics, back and forth*, IMA J. Num. Anal. (2014), dru010.
- [20] R. P. Feynman. *The principle of least action in quantum mechanics*, Ph.D thesis, Princeton University, Princeton, NJ, 1942.
- [21] P. J. Fitzsimmons. *Even and odd continuous additive functionals*, in *Dirichlet forms and stochastic processes (Beijing, 1993)*, de Gruyter, Berlin (1995), 139–154.
- [22] M. Freidlin. *Functional integration and partial differential equations*, Princeton University Press, Princeton, NJ, 1985.
- [23] M. Fukushima. *Dirichlet spaces and strong Markov processes*, Trans. Amer. Math. Soc., **162** (1971), 185–224.
- [24] M. Fukushima. *On a decomposition of additive functionals in the strict sense for a symmetric Markov process*, in *Dirichlet forms and stochastic processes (Beijing, 1993)*, de Gruyter, Berlin (1995), 155–169.
- [25] M. Fukushima, Y. Ōshima and M. Takeda. *Dirichlet forms and symmetric Markov processes*, Walter de Gruyter & Co., Berlin, 1994.
- [26] M. Fukushima and M. Tomisaki. *Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps*, Probab. Theory Related Fields, **106** (1996), 521–557.
- [27] A. Gloria and J.-C. Mourrat. *Quantitative version of the Kipnis-Varadhan theorem and Monte Carlo approximation of homogenized coefficients*, Ann. Appl. Probab., **23** (2013), 1544–1584.
- [28] A. Gloria, S. Neukamm and F. Otto. *Quantification of ergodicity in stochastic homogenization: optimal bounds via spectral gap on Glauber dynamics*, Inventiones mathematicae (2013), 1–61.
- [29] A. Gloria and F. Otto. *Quantitative results on the corrector equation in stochastic homogenization*, Preprint, arXiv:1409.0801 (2014).
- [30] J. A. Griepentrog and L. Recke. *Linear elliptic boundary value problems with non-smooth data: normal solvability on Sobolev-Campanato spaces*, Math. Nachr., **225** (2001), 39–74.
- [31] P. Hsu. *Probabilistic approach to the Neumann problem*, Comm. Pure Appl. Math., **38** (1985), 445–472.

- [32] P. Hsu. *On the Poisson kernel for the Neumann problem of Schrödinger operators*, J. London Math. Soc. (2), **36** (1987), 370–384.
- [33] K. Itô and H. P. McKean Jr.. *Diffusion processes and their sample paths*, Springer-Verlag, Berlin-New York, 1974.
- [34] M. Kac. *On distributions of certain Wiener functionals*, Trans. Amer. Math. Soc., **65** (1949), 1–13.
- [35] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, Springer-Verlag, New York, 1991.
- [36] R. S. Keskin and M. D. Grigoriu. *A probability-based method for calculating effective diffusivity coefficients of composite media*, Probabilist. Eng. Mech., **25** (2010), 249–254.
- [37] I. C. Kim, D. Cule and S. Torquato. *Comment on “Walker diffusion method for calculation of transport properties of composite materials”*, Phys. Rev. E, **61** (2000), 4659–4660.
- [38] I. C. Kim and S. Torquato. *First passage time calculation of the conductivity of continuum models of multiphase composites*, Phys. Rev. A, **43** (1991), 3198–3201.
- [39] I. C. Kim and S. Torquato. *Effective conductivity of suspensions of spheres by Brownian motion simulation* J. Appl. Phys., **69** (1991), 2280–2289.
- [40] C. Kipnis and S. R. S. Varadhan. *Central limit theorem for additive functional of reversible Markov processes and applications to simple exclusion*, Commun. Math. Phys., **104** (1986), 1–19.
- [41] A. Klenke. *Probability theory*, Springer, London, 2014.
- [42] A. Lejay. *Homogenization of divergence-form operators with lower-order terms in random media*, Probab. Theory Related Fields, **120** (2001), 255–276.
- [43] A. Lejay and S. Maire. *New Monte Carlo schemes for simulating diffusions in discontinuous media*, J. Comput. Appl. Math., **245** (2013), 97–116.
- [44] J.-C. Mourrat. *Variance decay for functionals of the environment viewed by the particle*, Ann. Inst. Henri Poincaré Probab. Stat., **47** (2011), 294–327.
- [45] J. Nash. *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math., **80** (1958), 931–954.
- [46] R. Nittka. *Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains*, J. Differential Equations, **251** (2011), 860–880.
- [47] G. C. Papanicolaou. *Diffusion in random media*, Surveys Appl. Math., **1** (1995), 205–253.
- [48] G. C. Papanicolaou and S. R. S. Varadhan. *Boundary value problems with rapidly oscillating random coefficients*, in *Random fields, Vol. I, II (Esztergom, 1979)*, 835–873, North-Holland, Amsterdam, 1981.
- [49] V. G. Papanicolaou. *The probabilistic solution of the third boundary value problem for second order elliptic equations*, Probab. Theory Related Fields, **87** (1990), 27–77.
- [50] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.
- [51] A. Piatnitski and E. Remy. *Homogenization of elliptic difference operators*, SIAM J. Math. Anal., **33** (2001), 53–83.
- [52] R. Rhodes. *Stochastic homogenization of reflected stochastic differential equations*, Electron. J. Probab., **15** (2010), 989–1021.
- [53] A. Rozkosz. *On a decomposition of symmetric diffusions with reflecting boundary conditions*, Stochastic Process. Appl., **103** (2003), 101–122.
- [54] A. Rozkosz and L. Słomiński. *Stochastic representation of reflecting diffusions corresponding to divergence form operators*, Studia Math., **139** (2000), 141–174.
- [55] M. Simon. *Bayesian anomaly detection in heterogeneous media with applications to geophysical tomography*, to appear in *Inverse Problems*.
- [56] N. A. Simonov and M. Mascagni. *Random Walk Algorithms for Estimating Effective Properties of Digitized Porous Media*, Monte Carlo Meth. and Appl., **10** (2004), 599–608.
- [57] E. Somersalo, M. Cheney and D. Isaacson. *Existence and uniqueness for electrode models for electric current computed tomography*, SIAM J. Appl. Math., **52** (1992), 1023–1040.
- [58] D. W. Stroock. *Diffusion semigroups corresponding to uniformly elliptic divergence form operators*, in *Séminaire de Probabilités, XXII*, Springer, Berlin (1988), 316–347.
- [59] H. Tanaka. *Homogenization of diffusion processes with boundary conditions*, Stoc. Anal. Appl., Adv. Probab. Related Topics, **7** (1984), 411–437.
- [60] S. Torquato. *Random heterogeneous materials*, Springer-Verlag, New York, 2002.

- [61] S. Torquato, I. C. Kim and D. Cule. *Effective conductivity, dielectric constant, and diffusion coefficient of digitized composite media via first-passage-time-equations*, J. Appl. Phys., **85** (1999), 1560–1571.
- [62] G. M. Troianiello. *Elliptic differential equations and obstacle problems*, Plenum Press, New York, 1987.
- [63] T. Zhang. *A probabilistic approach to Dirichlet problems of semilinear elliptic PDEs with singular coefficients*, Ann. Probab., **39** (2011), 1502–1527.
- [64] V. V. Žikov, S. M. Kozlov, O. A. Oleĭnik. *Homogenization of differential operators and integral functionals*, Springer-Verlag, Berlin, 1994.

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